

Network Coded Modulation with Partial Side-Information and Hierarchical Decode and Forward Relay Sharing in Multi-Source Wireless Network

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Abstract—We present a strategy for the relay sharing in the 2-source relay wireless network with the hierarchical decode and forward relaying. It processes both incoming coded symbols through the minimal hierarchical symbol mapping of the same cardinality as the two individual symbol streams and requires a full (perfect) Complementary Side-Information (C-SI) on the other stream available at the destination. This scheme is known to have an advantage in a *rectangular* shape capacity region which provides *higher sum-capacity* than the classical MAC. We investigate the capacity region of the case where the destination has only partial C-SI available through a wireless link. We conclude that the overall finite alphabet constrained capacity region including both MAC and BC phase remains *rectangular*. The BC phase region is limited by the C-SI link capacity and we conjecture that it obeys (in some cases with some small margin) the capacity product law of the individual relay to destination and C-SI links.

I. INTRODUCTION

A. Background And Related Work

Multi-node and multi-source wireless communication scenarios are currently under intensive investigation in the research community. Generally, these can be seen as similar to the Network Coding (NC) paradigm [1]. NC operates with a discrete (typical binary) alphabet over loss-less discrete channels. It is in fact an operation on data rather than on the channel codewords. NC has great potential in substantially increasing the throughput of complicated communication networks. An extension of these principles into the wireless (signal space) domain is however non-trivial. Some attempts have been carried out using a simple concatenation of NC and a single-link physical layer modulation and coding technique. This has number of drawbacks and only limited optimality. An optimal solution is a direct signal space domain code synthesis.

Limited code design and capacity region results are available for the simplest possible scenario of the 2-Way Relay Channel (2-WRC). Authors of [2], [3], [4], [5] concentrate on the distance optimization of the relay hierarchical mapping regions as

a function of the channel parametrization. A specific approach using a lattice code construction is taken by [6] and [7]. The authors of [6] and [7] provide lattice based code construction using principles of [8] but do not investigate the impact of the channel parametrization. The results of this paper are also based on our earlier work [9], [10], [11].

B. Concept of Sharing PHY Layer in Multi-Source and Multi-Node Networks

The research target is a general *multi-source* and *multi-node* wireless communication network where all network nodes (relays) contribute to providing data flows *without* an explicit routing. It can be described as a flood of the information having different “colors” (from individual sources) at the inputs. The network processes the “rainbow” mixture-color not distinguishing individual sources. The destinations pick a particular “color” from the received flood with the help of various forms of the side-information on other data streams available at the destination. Since the mixture-data flow represents jointly (but not necessarily individually distinguished) data stream, we call those data *hierarchical data* and corresponding relay processing a *hierarchical decode and forward* relay strategy.

This resembles the principles of Network Coding, however with important differences. (1) The information transfer is achieved through *signal-space wireless links* including all phenomena of *channel parametrization, fading and received signal superpositions*. (2) The distributed source coding joint with distributed channel coding can play an important role. The source-channel separation theorem does not hold generally for correlated sources. This problem arises mainly when we do not fully decode and re-code the hierarchical codewords at relays (e.g. due to the latency constraints). Then the resulting per-symbol relay processing must source-code the soft-symbol metric. These metrics are possibly correlated due to the codeword structure and due to the multi-path network flow topology.

The above stated concept relies on following tools.

- Network Coded Modulation is the network structure aware modulation and coding. Each node, either source or relay, transmits the signal that it can be processed in the remaining receiving nodes performing various joint relaying strategies.

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- Relaying strategies should work on the hierarchical data using hierarchical decode and forward strategy, not on individual separated data streams.
- Combined distributed source and channel coding at various levels including full frame and also per-symbol soft-decoding metric for symbol-wise relay strategies can help improving the overall network efficiency.

The simplest particular network example capable of demonstrating some of the above stated concepts is the 2-Source Relay Network. We will investigate some of its properties in this paper.

C. Goals of this Paper and Contributions

An outline of the paper stands on following points.

- 1) We present the strategy of the *relay sharing* by means of the *network structure aware* modulation and coding (Network Coded Modulation (NCM)).
- 2) We use a concept of the *Hierarchical Decode and Forward (HDF)* strategy ([9], [10], [11], [5]) at the relay. It processes both incoming coded symbols through the minimal hierarchical symbol mapping of the same cardinality as the two individual symbol streams. This scheme is known to have an advantage in a *rectangular* shape capacity region which provides *higher sum-capacity* than the classical MAC.
- 3) The Minimal HDF processing requires a full (perfect) Complementary Side-Information (C-SI) on the other stream. However this is not available at the destination. Only a *partial (imperfect) C-SI* is available through the wireless observation.
- 4) We concentrate on the BC phase and the impact of partial C-SI. The MAC phase is analyzed in [11].

The paper provides the following results and contributions.

- 1) We analyze the *capacity region degradation* of the HDF BC phase caused by various level of the partial C-SI knowledge.
- 2) All capacity regions are *finite alphabet constrained* (imposed by a practical implementation constraints).
- 3) The analysis of the hierarchical BC capacity lead to the product capacity law conjecture.

II. SYSTEM MODEL AND DEFINITIONS

A. 2-Source Relay Network with Partial C-SI

A 2-Source Relay Network (2-SRN) consists of two independent data sources SA and SB which want to deliver their data to two destinations DA and DB through shared relay R. During the first (MAC) stage sources A and B transmit to the relay and also destination B collects C-SI on the data A and destination A C-SI on the data B. The relay operates with HDF strategy. In the second (BC) stage, R produces on its output hierarchical symbols (details will be given later) carrying both data A and B only through the Hierarchical eXclusive Code (HXC). Both destination receive this common signal and combine that with the C-SI received in the MAC stage to decode the data. See Fig. 1. The figure also shows

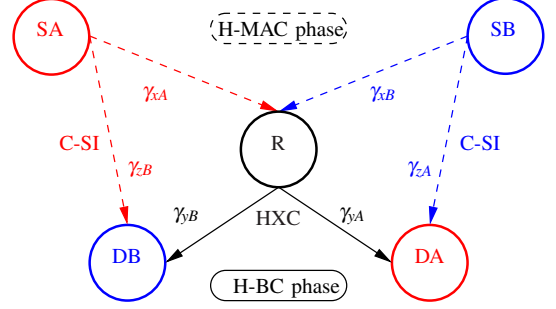


Figure 1. 2-Source Relay Network with Partial C-SI.

individual Signal-to-Noise Ratios (SNRs) of the individual links.

The system is symmetric in its structure ($SA \rightarrow R \rightarrow DA$, and $SB \rightarrow R \rightarrow DB$), which allows us to investigate it from the *perspective* of the *data flow A* with data B being only a nuisance parameter. All conclusions drawn for A should then be equally applicable to B.

Next sections define formal details. Subscripts A and B denotes variable associated with node A and B respectively. For notational simplicity, we omit the sequence number indices of individual symbols. We use also a slightly relaxed notation for the symbol and the codebook sets and the corresponding mapping and encoding operations. The notation uses the same letter symbol in two different contexts. This simplifies the reader's appreciation of the relatively large number of codebooks/alphabets involved. As an example, the codebook (the set of all codewords) is denoted by \mathcal{C} and it is optionally supplemented by a subscript corresponding to the particular codeword. The notation $\mathcal{C}(\mathbf{d})$ denotes a codeword corresponding to the data input \mathbf{d} . A similar notation holds for symbol alphabets, denoted by \mathcal{A} , and corresponding mapping operations $\mathcal{A}(\cdot)$. All continuous-valued time-domain signals are represented through their signal space (with an *orthonormal* basis) representations.

B. Exclusive Law and Hierarchical Exclusive Code/Alphabet

The *exclusive law* [3] is defined as a property of the joint representation of two data symbols through the function $\mathcal{X}_d(\mathbf{d}_A, \mathbf{d}_B)$. It must hold that

$$\mathcal{X}_d(\mathbf{d}_A, \mathbf{d}_B) \neq \mathcal{X}_d(\mathbf{d}'_A, \mathbf{d}_B), \forall \mathbf{d}_A \neq \mathbf{d}'_A, \quad (1)$$

$$\mathcal{X}_d(\mathbf{d}_A, \mathbf{d}_B) \neq \mathcal{X}_d(\mathbf{d}_A, \mathbf{d}'_B), \forall \mathbf{d}_B \neq \mathbf{d}'_B. \quad (2)$$

The functions fulfilling exclusive law will be called *exclusive mapping* and denoted by the operator $\mathcal{X}_d(\cdot, \cdot)$ specifying also the corresponding symbol (in this case d) for which it applies. Simply speaking, the exclusive mapping allows an inverting the mapping function provided we have a side-information on one of the data symbols. Assuming that the destination node B has perfect Side Information (SI) on the node's own data \mathbf{d}_B it can then decode the message \mathbf{d}_A (and similarly for node A). Data \mathbf{d}_B will be called *complementary data* from the perspective of the data \mathbf{d}_A operations. The SI on the

complementary data will be denoted as Complementary SI (C-SI).

The exclusive mapping can be applied at various levels — complete or partial codewords/messages or individual symbols. We define Hierarchical eXclusive Code (HXC) as a two-source codebook fulfilling the exclusive law. Similarly we define Hierarchical eXclusive Alphabet (HXA) when it is applied symbol-wise.

Assume a HXC codebook \mathcal{C}_{AB} with mapping $\mathcal{X} : (\mathcal{C}_A, \mathcal{C}_B) \mapsto \mathcal{C}_{AB}$. In order for the exclusive law to hold, the relay hierarchical codebook cardinality must satisfy $|\mathcal{C}_{AB}| \geq \max(|\mathcal{C}_A|, |\mathcal{C}_B|)$. On the other hand, the cardinality is upper-bounded by all possible \mathbf{d}_A and \mathbf{d}_B message combinations $|\mathcal{C}_{AB}| \leq |\mathcal{C}_A| |\mathcal{C}_B|$. The lower bound case requires perfect C-SI on the complementary data at the destination. The upper bound is in fact a classical MAC channel with joint and separate decoding of both data messages at the relay and requires *no* C-SI. Any situation in between those two extreme cases requires partial C-SI at the destination.

The HXC having the minimal cardinality $|\mathcal{C}_{AB}| = \max(|\mathcal{C}_A|, |\mathcal{C}_B|)$ will be called Minimal-HXC (M-HXC). This case puts minimum *throughput* requirements on the BC channel at the price of requiring perfect C-SI at the destination. All codebooks with a higher cardinality will be called Extended-HXC (E-HXC).

C. H-MAC Phase

Source data messages are $\mathbf{d}_A, \mathbf{d}_B$ and they are composed of data symbols $d_A, d_B \in \mathcal{A}_d = \{0, 1, \dots, M_d - 1\}$, with alphabet cardinality $|\mathcal{A}_d| = M_d$. Source node codewords are $\mathbf{c}_A, \mathbf{c}_B$ with code symbols $c_A, c_B \in \mathcal{A}_c$, $|\mathcal{A}_c| = M_c$. The encoding operation is performed by the encoders $\mathcal{E}_A, \mathcal{E}_B$ with codebooks $\mathbf{c}_A \in \mathcal{C}_A$ and $\mathbf{c}_B \in \mathcal{C}_B$. Transmitted channel symbols are $s_A = s(c_A)$, $s_B = s(c_B)$, $s_A, s_B \in \mathcal{A}_s \subset \mathbb{C}^N$. We assume a common channel symbol mapper $\mathcal{A}_s(\cdot)$. A signal space representation of the overall coded frame is $\mathbf{s}_A(\mathbf{c}_A)$ and $\mathbf{s}_B(\mathbf{c}_B)$.

The received signal at the relay in a parametrized flat fading (constant over the frame) channel is

$$u = s_A + h s_B. \quad (3)$$

The relative equivalent parametrization by $h \in \mathbb{C}$ represents the parametric channel with both parametric links $u' = h_A s_A + h_B s_B$ by a proper rescaling by $1/h_A$ and denoting $h = h_B/h_A$, $h_A, h_B \in \mathbb{C}^1$. The received signal at the relay is

$$x = u + w \quad (4)$$

where the circularly symmetric complex Gaussian noise w has the variance σ_w^2 per complex dimension. See Fig. 2.

The SNR is defined as the ratio of the real base-band symbol energy of one source (e.g. A, to have a fair comparison for reference cases) to the noise power spectrum density ratio $\gamma_{xA} = (\mathcal{E}_{s_A}/2)/N_0$. Assuming orthonormal basis signal space complex envelope representation of the AWGN, we have $\sigma_w^2 = 2N_0$ and thus $\gamma_{xA} = \mathbb{E}[\|s_A\|^2]/\sigma_w^2$.

D. H-BC Phase

The HDF front-end (H-MAC phase) processing at the relay produces the hierarchical symbols $\mathbf{d}_{AB} = \mathcal{X}_d(\mathbf{d}_A, \mathbf{d}_B)$ which are the *minimal* HXC. The data \mathbf{d}_{AB} are encoded at the back-end (H-BC phase) of the relay to produce codewords $\mathbf{b}_{AB} = \mathcal{E}_R(\mathbf{d}_{AB})$. The discrete code symbols are mapped in symbol-by-symbol manner on the signal space symbols $v = v(b_{AB}) \in \mathcal{A}_v$ and broadcast to destinations DA and DB. At the destination for data A, the received signal space symbols are

$$y_A = v(b_{AB}) + w_A \quad (5)$$

where the complex circularly symmetric AWGN w_A has variance σ_A^2 per complex dimension. We denote the signal space symbols at node A (a destination for data B) similarly $y_B = v + w_B$. We define the SNR on the R to DA link as $\gamma_{yA} = \mathbb{E}[\|v\|^2]/\sigma_{w_A}^2$.

The C-SI on the data from the source SB are received at the destination DA during the first H-MAC phase and they are stored for the later processing at the second H-BC phase. The transmitted signal is identical to the one transmitted at H-MAC. The received signal is

$$z_A = s(c_B) + \xi_A \quad (6)$$

where the complex circularly symmetric AWGN ξ_A has variance $\sigma_{\xi_A}^2$ per complex dimension. The SNR of the C-SI link is $\gamma_{zA} = \mathbb{E}[\|s_A\|^2]/\sigma_{\xi_A}^2$. Fig. 3.

III. HIERARCHICAL DECODE AND FORWARD STRATEGY

A. Relay Sharing with Perfect Complementary Data Side Information

The relay sharing with perfect C-SI at the destinations is based on the Hierarchical Decode and Forward (HDF) relaying strategy. The relay processes only hierarchical symbols which jointly represent both data streams but individual data can be extracted only on condition of providing C-SI at the destination.

In the case of the *minimal* HXC, it can be shown ([9], [10]) that the H-MAC capacity region is rectangular and provides a significant performance advantage over the classical MAC channel sharing. The minimal HXC requires a perfect C-SI at the destination in order not to degrade the HDF relay performance. If that perfect C-SI is available then the overall system performance is dominated by the H-MAC stage in a typical scenario with a system with a symmetric topology (i.e. $\gamma_x \approx \gamma_y$). The relay transmits a *common* hierarchical data stream for both destinations and thus the BC phase has an easier situation in a comparison with the MAC phase.

The hierarchical data can be viewed as the network coded data streams. But notice an important difference. When we simply concatenate independently the NC and then the channel coding, the data generated at the relay are obtained by jointly decoding both individual streams from A and B. The NC mainly serves for the BC stage transmission. The MAC phase is classical joint decoding. In our hierarchical approach this

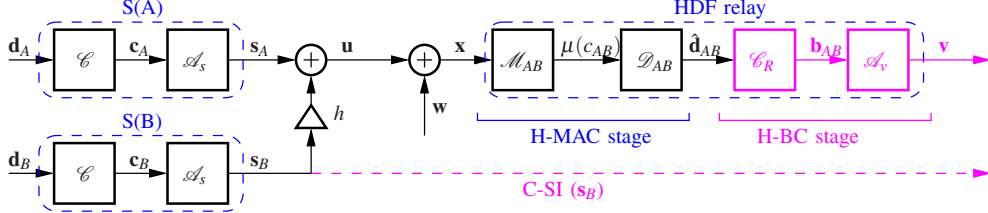


Figure 2. The system model for H-MAC phase with layered HXC design.

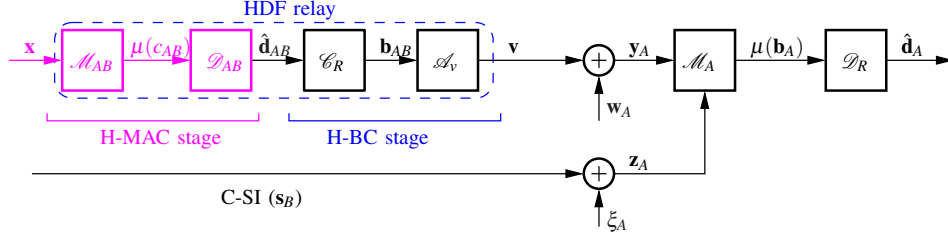


Figure 3. The system model for H-BC phase with layered HXC design.

does not happen. The hierarchical data are directly obtained from the received signal observations without the necessity of individual source data decoding. The MAC stage encoding must be such that the observation at the relay allows HXC mapping to the hierarchical data, which may then be encoded into \mathbf{v} . In other words, alongside the *complete signal path* (MAC and BC), the coding *must always be HXC* w.r.t. the *hierarchical data*.

B. Layered Coding with Symbol-wise Decoding Metric

The layered design relies on the theorems described in detail in paper [9] and also in [10]. Here, we only state the results. The layered HXC design is possible and the capacity is achieved by an outer standard channel code $\mathcal{C}_A = \mathcal{C}_B = \mathcal{C}$ combined with inner symbol-wise HXA \mathcal{A}_s . We will refer to this as the *layered HXC design theorem*. The theorem assumes the minimal HXA mapping for the data and generally the multi-symbol HXA mapping for constellation space symbols received by the relay $c_{AB} = \mathcal{X}_c(c_A, c_B)$ where $u(c_A, c_B) = \mathcal{A}_u(c_{AB})$. The capacity is alphabet constrained by *H-MAC channel hierarchical symbols* and the H-MAC rate region has a rectangular shape

$$R_A \leq C_{AB}, R_B \leq C_{AB}, C_{AB} = I(c_{AB}; x). \quad (7)$$

The mutual information $I(c_{AB}; x)$ is not a capacity in a strict sense since we do not maximize over the input distribution. We will however assume that this distribution is given and it is a uniform distribution of a discrete channel input. This will be called a *uniform-input (alphabet constrained) capacity*. We denote it by the letter C however the uniform-input assumption must be kept in mind.

There is also an additional advantage of the layered design. The exclusive property at the symbol level allows a simple determination of the soft *per-symbol* measure decoding metric $\mu(c_{AB})$ for hierarchical symbols at the relay. Figs. 2 and 3 show the layered processing structure.

IV. HIERARCHICAL BC WITH PARTIAL SIDE INFORMATION

A. H-BC with Minimal HXC and Partial C-SI

1) Invertibility of Minimal Exclusive Mapping:

Lemma 1 (Reciprocal minimal exclusive mapping):

Assume a *minimal exclusive mapping* $a_{AB} = \mathcal{X}(a_A, a_B)$ for finite alphabet symbols a_A, a_B, a_{AB} with the cardinality M . Then we can always construct some other *minimal exclusive mapping* \mathcal{X}' such that $a_A = \mathcal{X}'(a_{AB}, a_B)$.

Proof: For a finite alphabet, we can always index the symbols by integers $i_A, i_B, i_{AB} \in \{0, \dots, M-1\}$. The minimal exclusive mapping for the index alphabets can be implement by mod M operations $i_{AB} = i_A \oplus i_B$. But it directly implies that $i_A = i_{AB} \ominus i_B$ which also forms a minimal exclusive mapping. ■

This property simply means that given the knowledge of arbitrary 2 symbols out of a_A, a_B, a_{AB} , we can uniquely obtain the remaining one provided that the mapping is the minimal one.

2) *Symbol-wise Destination Decoding Metric:* Assume a layered HXC with *common* H-MAC and H-BC relay code $\mathcal{C}_R = \mathcal{C}$ and transmitting the symbols at relay

$$\mathbf{v} = \mathbf{v}(b_{AB}), \mathbf{b}_{AB} = \mathcal{C}(\mathbf{d}_{AB}) \quad (8)$$

where symbol-wise mappings $d_{AB} = \mathcal{X}_d(d_A, d_B)$ and $b_{AB} = \mathcal{X}_b(b_A, b_B)$ are *minimal exclusive mappings*. The destination received signal symbols are $\mathbf{y}_A = \mathbf{y}_A(b_{AB})$. The function $\mathbf{y}_A(b_{AB})$ represents a *general* input-output description of the H-BC observation at the destination including full channel model. An example is a special case of AWGN R to DA channel $\mathbf{y}_A = \mathbf{y}_A(b_{AB}) = \mathbf{v}(b_{AB}) + \mathbf{w}_A$.

Assume a C-SI transmitted signal *sharing* the same codebook as the H-MAC phase $s_B = s(c_B)$, $\mathbf{c}_B = \mathcal{C}(\mathbf{d}_B)$. The C-SI is in fact the same signal as it is transmitted from the source at the H-MAC phase. The destination received signal

symbols are $z_A = z_A(c_B)$ which is conditionally independent with observation y_A : $z_A \perp y_A | b_{AB}, c_B$. A general input-output description of the C-SI link is represented by $z_A(c_B)$.

From lemma 1, it must hold $d_A = \mathcal{X}'_d(d_{AB}, d_B)$. The symbols b and c are coded by the same codebook, thus they must share the same alphabet. Again, from the lemma 1, it holds $b_A \equiv c_A = \mathcal{X}'_c(b_{AB}, c_B)$ and also $b_B \equiv c_B$. The symbol equivalence comes from the common isomorphic indexing. The distributive law of the coding over the minimal per-symbol mapping (see [9] or [10]) gives

$$\mathcal{X}'_c(b_{AB}, c_B) = \mathcal{C}(\mathcal{X}'_d(d_{AB}, d_B)). \quad (9)$$

Therefore, d_A must be decodable from the observations $y_A = y_A(b_{AB})$ and $z_A = z_A(c_B)$.

The symbol-wise decoding metric $\mu(b_A) = p(y_A, z_A | b_A)$ is

$$\begin{aligned} p(y_A, z_A | b_A) &= \sum_{c_B} p(y_A, z_A | b_A, c_B) p(c_B) \\ &= \sum_{c_B} p(y_A | b_A, c_B) p(z_A | b_A, c_B) p(c_B) \\ &= \sum_{c_B} p(y_A | \mathcal{X}'_c(b_A, c_B)) p(z_A | c_B) p(c_B). \end{aligned} \quad (10)$$

This metric is used by the destination decoder \mathcal{D} corresponding to the coded \mathcal{C} .

B. Equivalent H-BC Channel

The destination B has two channel observations y_A and C-SI link z_A . The equivalent channel for the capacity evaluation has multi-output form

$$[y_A, z_B] = [y_A(b_{AB}), z_A(c_B)]. \quad (11)$$

C. H-BC Symmetric Capacity with Partial C-SI

The H-BC phase transmits a common HXC signal to both destinations. The H-BC capacity region is thus *rectangular* with maximum rates given by $I(b_A; y_A, z_A)$ and similarly for destination B.

The symmetric alphabet constrained H-BC capacity corresponds to the mutual information evaluated for the uniform code symbols distribution

$$C_{\text{HBC}} = I(b_A; y_A, z_A) = \mathcal{H}[y_A, z_A] - \mathcal{H}[y_A, z_A | b_A]. \quad (12)$$

Both entropies are relatively easy to evaluate

$$\mathcal{H}[y_A, z_A] = -\mathbb{E}_{y_A, z_A} [\lg p(y_A, z_A)], \quad (13)$$

$$\mathcal{H}[y_A, z_A | b_A] = -\mathbb{E}_{y_A, z_A, b_A} [\lg p(y_A, z_A | b_A)]. \quad (14)$$

For the Gaussian channels, it is

$$\begin{aligned} p(y_A, z_A | b_A) &= \\ &= \sum_{c_B} p_{w_A}(y_A - v(\mathcal{X}'_c(b_A, c_B))) p_{\xi_A}(z_A - s(c_B)) p(c_B) \end{aligned} \quad (15)$$

and

$$p(y_A, z_A) = \sum_{b_A} p(y_A, z_A | b_A) p(b_A) \quad (16)$$

where $p_{w_A}(\cdot)$ is the PDF of the R to DA link additive noise and $p_{\xi_A}(\cdot)$ is the PDF of the SB to DA side-information link additive noise (see Fig. 3).

For a comparison purposes, we also evaluate pure C-SI link capacity, as if there was no other signal at all. The capacity $C_{\text{CSI}} = I(c_B; z_A)$ is a simple capacity of the Gaussian alphabet constrained channel.

V. NUMERICAL RESULTS

All symmetric alphabet constrained capacities were evaluated numerically by the Monte-Carlo integral evaluation. The detailed procedure follows similar principles as explained in [9] and [10]. We assumed identical alphabets $\mathcal{A}_s = \mathcal{A}_v$. The H-BC capacity as a function of the γ_z SNR for various alphabets and relay-to-destination SNR γ_x are on Figs. 4, 5, 6, 7. We also show a pure “stand-alone” capacity of the C-SI link for a comparison.

A slightly different perspective on the the same results is shown on Figs. 8, 9, 10, 11. We plotted the H-BC capacity as function of the stand-alone C-SI link capacity *implicitly* parametrized by all γ_z . The curves are plotted for various alphabets and relay-to-destination link SNR.

VI. DISCUSSION AND CONCLUSIONS

The numerical results show some qualitative phenomena.

- 1) The H-BC capacity is upper bounded by the C-SI link capacity (Figs. 4, 5, 6, 7)

$$C_{\text{HBC}}(\gamma_y, \gamma_z) \leq C_{\text{CSI}}(\gamma_z), \quad \forall \gamma_y, \gamma_z. \quad (17)$$

This is not surprising, and in fact, it directly follows from the assumption of the *minimal* HXC used. Unless the C-SI link is capable providing reliable C-SI, we cannot decode the main data from the source A.

- 2) This cannot be exceeded even for the perfect relay-to-destination link $\gamma_y \rightarrow \infty$. Then the capacity is purely given by the C-SI capacity

$$C_{\text{HBC}}(\gamma_y \rightarrow \infty, \gamma_z) = C_{\text{CSI}}(\gamma_z). \quad (18)$$

This corresponds to the linear identity function in Figs. 8, 9, 10, 11.

- 3) However a rather surprising fact is that this linear law also holds for finite values of γ_y (see Figs. 8, 9, 10, 11)

$$C_{\text{HBC}}(\gamma_y, \gamma_z) = \alpha C_{\text{CSI}}(\gamma_z) \quad (19)$$

where

$$\alpha = \frac{C_{\text{HBC}}(\gamma_y, \gamma_z \rightarrow \infty)}{C_{\text{HBC}}(\gamma_y \rightarrow \infty, \gamma_z \rightarrow \infty)}. \quad (20)$$

We used the fact that the *perfect* C-SI case corresponds to $\gamma_z \rightarrow \infty$, with $C_{\text{perfCSI}} = \lg |\mathcal{A}_s|$ and $C_{\text{HBC}}^{\text{perfCSI}}(\gamma_y) = C_{\text{HBC}}(\gamma_y, \gamma_z \rightarrow \infty)$. The capacity $C_{\text{HBC}}^{\text{perfCSI}}$ is however given as a simple finite alphabet Gaussian link. The capacity

$$C_{\text{HBC}}(\gamma_y \rightarrow \infty, \gamma_z \rightarrow \infty) = \lg |\mathcal{A}_v| = \lg |\mathcal{A}_s| \quad (21)$$

corresponds to both perfect (error-less) relay-to-destination and C-SI links.

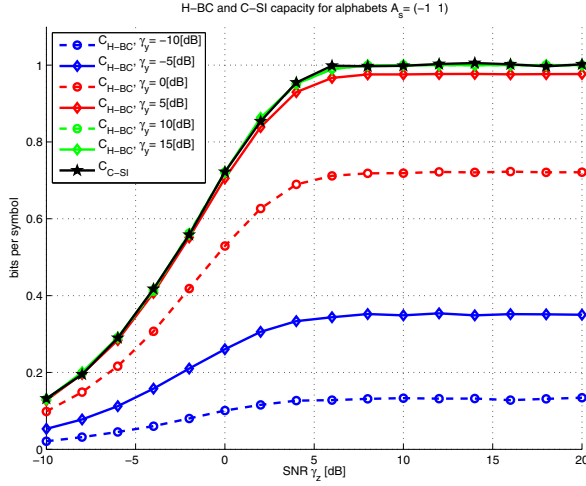


Figure 4. BPSK alphabet constrained capacity of H-BC stage and capacity of the complementary SI link.

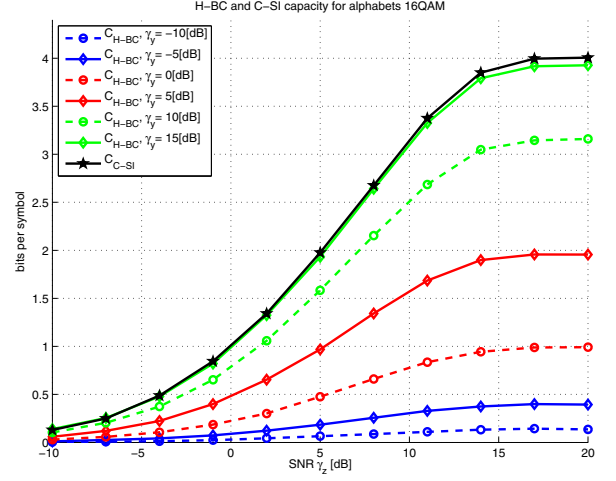


Figure 7. 16QAM alphabet constrained capacity of H-BC stage and capacity of the complementary SI link.

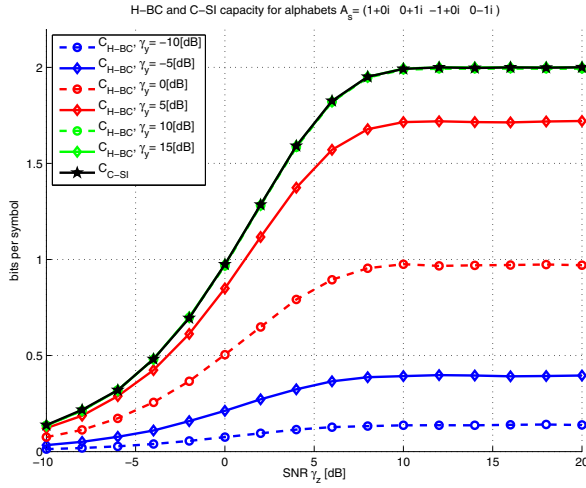


Figure 5. QPSK alphabet constrained capacity of H-BC stage and capacity of the complementary SI link.

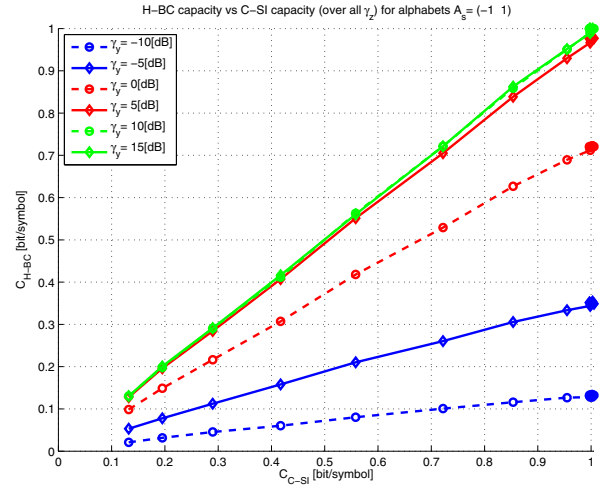


Figure 8. BPSK alphabet constrained capacity of H-BC versus capacity of C-SI link as a parametric plot over all γ_z .

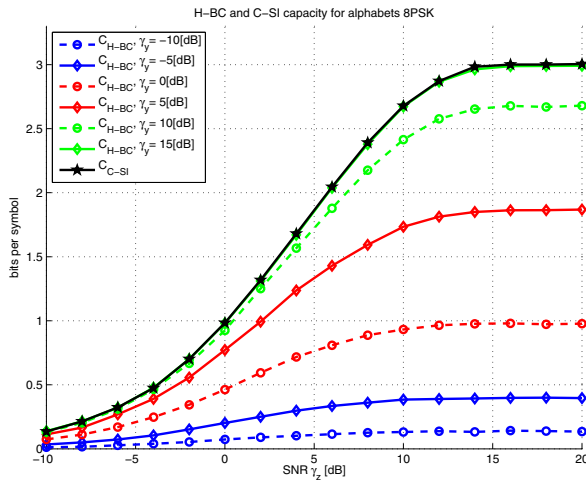


Figure 6. 8PSK alphabet constrained capacity of H-BC stage and capacity of the complementary SI link.

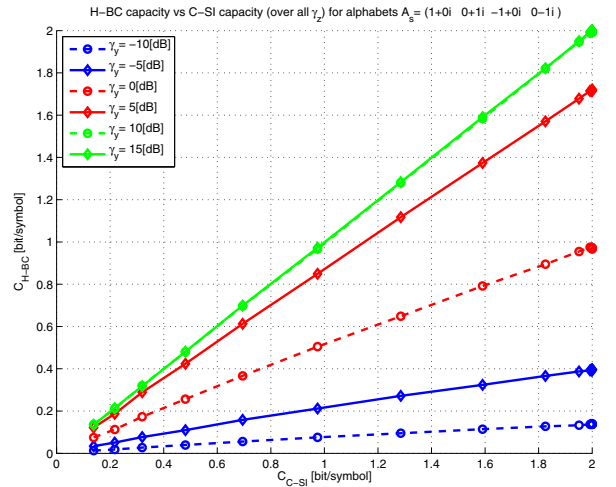


Figure 9. QPSK alphabet constrained capacity of H-BC versus capacity of C-SI link as a parametric plot over all γ_z .

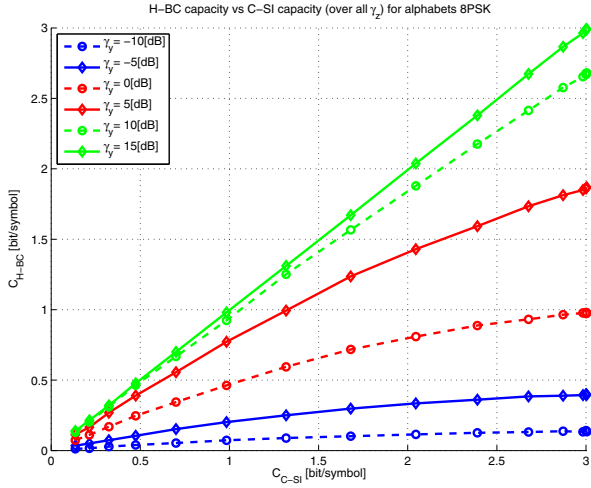


Figure 10. 8PSK alphabet constrained capacity of H-BC versus capacity of C-SI link as a parametric plot over all γ_z .

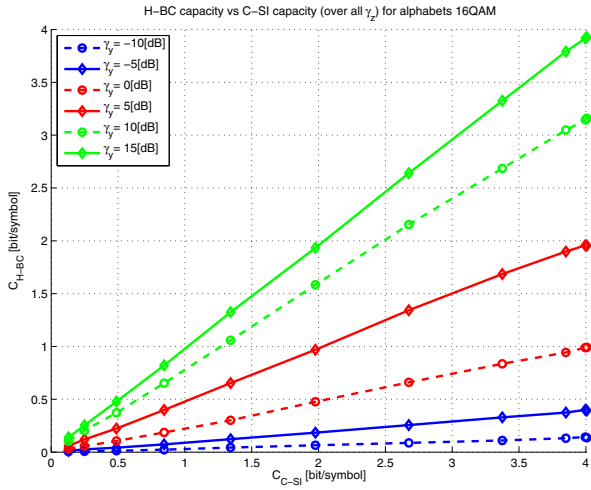


Figure 11. 16QAM alphabet constrained capacity of H-BC versus capacity of C-SI link as a parametric plot over all γ_z .

The above stated facts leads to the following conjecture.

Conjecture 1 (Product capacity law of the H-BC): The H-BC finite alphabet constrained capacity with minimal HXC and common source and relay alphabet \mathcal{A} is given by

$$C_{HBC}(\gamma_y, \gamma_z) = \frac{C_{HBC}^{\text{perfCSI}}(\gamma_y) C_{CSI}(\gamma_z)}{\lg |\mathcal{A}|}. \quad (22)$$

Note: The case of 8PSK exhibits very slight depart (a positive margin) from this law (see Fig. 10). An explanation of this phenomena is yet to be investigated.

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