Symmetric Capacity of Nonlinearly Modulated Finite Alphabet Signals in MIMO Random Channel with Waveform and Memory Constraints

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Abstract—**The paper shows a procedure for evaluating the symmetric ergodic capacity of nonlinearly modulated finite alphabet signals in MIMO random channel. We first investigate informationally equivalent signal space representation for finite alphabet signals. The signals are allowed to be multidimensional per one antenna to reflect the modulation nonlinearity. The informationally equivalent signal space representation allows a factorization per one eigenmode (depending purely on one marginal eigenvalue) in the capacity evaluation in MIMO channel. The evaluation complexity does not depend on the MIMO dimensions. We consider two types of channel signals constraints. (1) A constraint on the waveform only. (2) A constraint on the waveform and memory (modeled as a Markov chain). The mean symmetric capacity for the former one is derived. For the latter one, we derive lower and upper bounds based on the entropy rate approximation.**

I. Introduction

A. Motivation

The MIMO (Multiple-Input Multiple-Output) communication channel provides excellent capacity potential. However most of the research effort has been concentrated so far on the investigation of *linear* modulation schemes. The *nonlinear* ones remained somewhat at a marginal scope. Apparently this is caused by much more involved treatment. On the other side, the nonlinear modulations have number of attractive properties. It is mainly the resistance of the constant envelope modulations to the nonlinear distortion and more degrees of freedom for choosing the modulations waveforms in higher number of dimensions.

In this paper, we attack the mathematical difficulties associated with the nonlinearity by allowing a *finite multidimensional alphabet* channel symbols (waveforms). The finiteness assumption does not affect the usability of the results since any practically used scheme must use the finite alphabet anyway.

B. Goals

The goal of the paper is to evaluate the capacity of the MIMO Rayleigh channel with uncorrelated coefficients constrained on the input by the choice of particular *nonlinear* digital modulation. This imposes the constraint on the *waveform class* used, e.g. a constant envelope one. They are generally *multidimensional per one antenna* (consequence of nonlinearity) and are drawn from the *finite* set. Additionally we assume a constraint on the modulation *memory*. The memory controls the waveform choice in the symbol sequence. It can be used e.g. to guarantee the continuity of the phase (CPM). We want to demonstrate how (a) higher dimensionality of nonlinear signals and (b) the presence of the memory affect the capacity. The capacity evaluation should be factorisable in such a way that allows numerically feasible procedure *not* depending on the MIMO channel dimensions. No channel state information is assumed at the transmitter.

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C. Paper milestones

(1) We develop an equivalent signal space representation of a nonlinear *finite* alphabet modulation that allows factorization of the MIMO capacity evaluation per one eigenmode depending purely on *one marginalized* eigenvalue. This reduces the computational load when later evaluating the integrals in the capacity. It is based on a *simple to obtain* unitary decomposition (in contrast to Gram-Schmidt or Laurent expansion).

(2) We evaluate symmetric capacity for IID (Independent Identically Distributed) channel symbols—nonlinearly modulated waveforms with N_h dimensions per one antenna.

(3) We extend the previous case for channel symbols modeled as a Markov chain. The first order approximation of the lower and upper bound is derived. It is based on the entropy rates inequalities.

D. Distinguishing features of the paper in the context of the related work

In the contrast to a huge number of papers dealing with MIMO channel capacity for unconstrained linear modulation (represented by [1], [2]), the attention put on *nonlinear* and *waveform* and/or *memory* constrained modulation is much lower. The paper [3] deals with continuous valued constant envelope signals capacity in AWGN scalar channel. The paper [4] also assumes scalar channel and heavily relies on the Laurent decomposition of CPM modulation. The paper

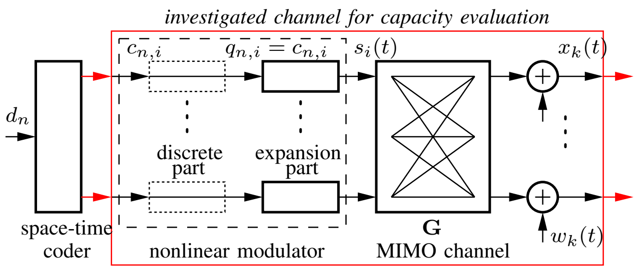

Fig. 1. System with constraint on waveform only.

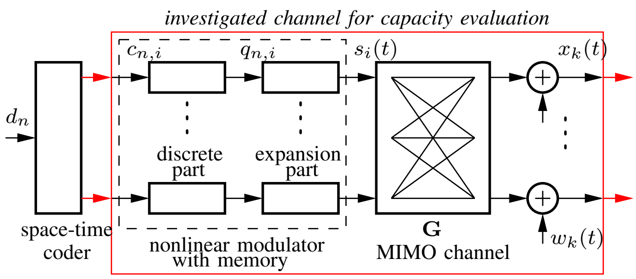

Fig. 2. System with constraint on waveform and memory.

[5] focuses only on linear modulations in MIMO channel and derives bounds that partially mask the dependence on particular constellation. The paper [6] concentrates only on linear modulations and partially extends the the results for modulation with memory. It builds on the same ground as [5]. Both [6] and [5] require averaging over the *complete MIMO channel matrix* which makes the evaluation rather difficult for higher number of antennas. None of the references address the problem of *nonlinear* (dimension per antenna >1) modulation with arbitrary *waveform* and *memory* constraint in *MIMO* channel. These problems are addressed in our paper. Moreover we provide the mean capacity evaluation in *factorized* form over one marginal eigenvalue allowing easy evaluation for high number of antennas.

II. System Model and Definitions

A. System setup for capacity investigation

We investigate the information capacity of a *composite channel* formed by a concatenated chain of *nonlinear* multiple antenna modulator and MIMO linear channel with AWGN. The nonlinear modulator is considered in two variants. **(a)** Memoryless nonlinear modulator (Fig. 1). The discrete part of the modulator is an identity mapping. This imposes a constraint only on the class of waveforms used (e.g. the constant envelope FSK type). **(b)** Nonlinear modulator with memory (Fig. 2). This situation imposes a constraint on both, the waveforms and the modulation memory. The additional constraint on memory can be used e.g. for controlling the continuity of the phase of constant envelope modulation (e.g. CPM type).

B. Modulator

1) Nonlinear modulator: A multiple antenna modulator produces *nonlinearly* modulated transmitted signal. The complex envelope signal on ith transmit antenna is

$$s_i(t) = \sum_n h(q_{n,i}, t - nT_S) \tag{1}$$

where T_S is the symbol period, $h(q,t)$ is generally *nonlinear* modulation function (waveform), $q_{n,i} \in \{q^{(m)}\}_{m=1}^{M_q}$ is a channel symbol and n is its sequence number. The channel symbol depends on the modulator input data $c_{n,i} \in \{c^{(m)}\}_{m=1}^{M_c}$ and modulator state $\sigma_{n,i} \in \{\sigma^{(m)}\}_{m=1}^{M_\sigma}$ through time-invariant generally *nonlinear* function $q_{n,i} = q_i(c_{n,i}, \sigma_{n,i})$. M_q, M_c, M_σ are alphabet (per-transmitter) sizes for channel symbol, input codeword, and modulator state respectively.

The function $q_i(c_{n,i}, \sigma_{n,i})$ forms a discrete part of the modulator and the function $h(q,t)$ forms an expansion part (discrete input, continuous waveform output) of the modulator. The set of all possible modulator expansion part output waveforms is $h(\centerdot, t) \in \{h^{(m)}(t)\}_{m=1}^{M_q}$. The modulation functions are assumed to be Nyquist ones. This means that the modulation memory is entirely located in the discrete part of the modulator—the expansion part does not introduce memory. Modulators (discrete and expansion parts) in individual transmitter branches are independent and identical.

2) Modulator constraint—waveform only, IID channel symbols: The constraint on the set of $h(\centerdot, t) \in \{h^{(m)}(t)\}_{m=1}^{M_q}$ available through the set of $q_{n,i} \in \{q^{(m)}\}_{m=1}^{M_q}$ is the *fundamental* one that controls the *class* of used *channel waveforms* (e.g. the constant envelope ones).

3) Modulator constraint—waveform and memory: The mapping $c_{n,i} \mapsto q_{n,i}$ puts an additional *optional* constraint related to the *memory* of modulation. It is useful to impose e.g. the continuity of the phase which could not be modeled otherwise. Assuming $(c_{n,i}, \sigma_{n,i}) \mapsto q_{n,i}$ being a *one-to-one* mapping, the sequence $\{q_{n,i}\}_n$ forms a *stationary* Markov chain with the transition matrix $\boldsymbol{\Pi}$ with elements $\Pi_{\ell,\ell'} = \Pr\{q_{1,i} = q^{(\ell')} | q_{0,i} = q^{(\ell)}\}$. The transition matrix can be obtained from the state equation $\sigma_{n+1,i} = \sigma(c_{n,i}, \sigma_{n,i})$ and a priori distribution $\{P_{m,i}\}_m = \{\Pr\{c_{n,i} = c^{(m)}\}\}_{m=1}^{M_c}$ of input symbols $c_{n,i}$ as $\Pr\{q_{1,i}|q_{0,i}\} = \Pr\{c_{1,i}, \sigma_{1,i}|c_{0,i}, \sigma_{0,i}\} = \Pr\{c_{1,i}\}\Pr\{\sigma_{1,i}|c_{0,i}, \sigma_{0,i}\}$ for all transmit antenna indices i.

C. Channel

The channel is assumed to be frequency flat block-constant fading (N_T, N_R) MIMO channel with AWGN (Additive White Gaussian Noise). A received signal at kth receive antenna is

$$x_k(t) = \sum_{i=1}^{N_T} g_{ki} s_i(t) + w_k(t) \tag{2}$$

where g_{ki} are channel coefficients and $\{w_k(t)\}_{k=1}^{N_R}$ are IID zero mean rotationally invariant complex white Gaussian noise processes with power spectrum density $S_w(f) = 2N_0$. Channel coefficients are zero mean IID complex Gaussian random variables with unity variance $\mathrm{E}[|g_{ki}|^2] = 1$. We also form a $(N_R \times N_T)$ channel matrix $[\mathbf{G}]_{ki} = g_{ki}$.

III. INFORMATIONALLY EQUIVALENT SIGNAL SPACE REPRESENTATION

A. Signal space

A continuous time domain system representation defined in Sec. II is not suitable for the capacity evaluation. A signal space (complex orthonormal (ON) basis) representation is much easier to handle—the AWGN representation becomes random vector with mutually uncorrelated components. This would not happen, if the base was not orthogonal. The signal space representation of the nonlinear modulated signal is somewhat more difficult to get in comparison with a linear modulation. In the case of *linear* modulation with Nyquist modulation impulse, the signal space corresponding to one channel symbol is *one-dimensional* (complex) and the modulation impulse can directly serve as an ON basis function.

Unlike the linear case, the continuous model of the nonlinear modulation (1) does not give a direct hint how to obtain the signal space representation. A dimension of the space *spanned by the modulation waveforms* for one symbol *per one antenna* $N_h = \dim(\operatorname{span}(\{h^{(m)}(t)\}_{m=1}^{M_q}))$ is generally $N_h \geq 1$ and $N_h \leq M_q$. It is important not to confuse the channel symbol dimensionality N_h (per one antenna) caused by the nonlinearity with the dimensionality given by multiple transmit antennas N_T. We can directly use the Gram-Schmidt process to get the ON basis. For a limited class of functions (binary CPM), the Laurent expansion is often used. However this expansion does *not* provide complex ON basis and the Gram-Schmidt process has to be applied anyway. Both these approaches are rather cumbersome and unnecessary.

B. Informationally equivalent multidimensional alphabets

It can be shown [7], using simple Euclidean space distance arguments, subspace projection, and the definition of the entropy, that arbitrary two *finite* multidimensional alphabets $\mathcal{A} = \{\mathbf{a}^{(i)}\}_{i=1}^{M} \subset \mathbb{C}^{n_a}$ and $\mathcal{B} = \{\mathbf{b}^{(i)}\}_{i=1}^{M} \subset \mathbb{C}^{n_b}$ with the dimension of spanned subspaces $\dim(\operatorname{span}(\mathcal{A})) = \dim(\operatorname{span}(\mathcal{B})) \leq \min(n_a, n_b)$ are mutually *informationally equivalent if* they have the same Gram matrix (correlation matrix) $\mathbf{R} = \mathbf{A}^H\mathbf{A} = \mathbf{B}^H\mathbf{B}$ where $\mathbf{A} = [\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(M)}]$ and $\mathbf{B} = [\mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(M)}]$. Superscript $(.)^H$ denotes Hermitian transposition. The equivalence means that they can be used one instead the other providing the same mutual information in the channel with additive noise. Such alphabets have *identical distances* between member pairs $\forall i, k : \|\mathbf{a}^{(i)} - \mathbf{a}^{(k)}\| = \|\mathbf{b}^{(i)} - \mathbf{b}^{(k)}\|$. Actual dimensions of vectors n_a and n_b can be arbitrary. Important is the *equivalence of the dimension* of the *spanned* subspaces.

This general result helps us to easily generate informationally equivalent signal space representation of modulated signals (L_2 space alphabet $\{h^{(i)}(t)\}_i$) based on the Gram matrix equivalence. We first compute the Gram matrix from the continuous domain definition of the modulation functions. Its elements are $[\mathbf{R}_h]_{i,k} = \int_{-\infty}^{\infty} h^{(i)}(t)h^{(k)*}(t)\,dt,\ i,k \in \{1, \ldots, M_q\}$. The informationally equivalent signal space representation of these functions organized column-wise into $(N_h \times M_q)$ matrix is $\mathbf{S} = [\mathbf{s}^{(1)}, \ldots, \mathbf{s}^{(M_q)}]$. The corresponding Gram matrix is $\mathbf{R_s} = \mathbf{S}^H\mathbf{S}$ and it must hold $\mathbf{R_s} = \mathbf{R}_h$. The matrix $\mathbf{R}_h$ can be factorized using unitary decomposition

$$\mathbf{R}_h = \mathbf{F}\mathbf{K}\mathbf{F}^H = \mathbf{F}\mathbf{K}^{1/2}\mathbf{K}^{1/2}\mathbf{F}^H. \tag{3}$$

$\mathbf{F}$ is $(M_q \times N_h)$ matrix with orthonormal columns. For $N_h < M_q$, we remove the columns corresponding to zero eigenvalues from the unitary matrix. $\mathbf{K}$ is $(N_h \times N_h)$ diagonal matrix with *nonzero* eigenvalues of $\mathbf{R}_h$. Clearly, the *informationally equivalent signal space representation* of the transmitted signals (composed as columns into a matrix) is $\mathbf{S} = \mathbf{K}^{1/2}\mathbf{F}^H$.

C. Signal space system model

From now on, we denote the *equivalent* signal space expansion of the signal at ith transmit antenna at sequence number n as $\mathbf{s}_{n,i} \in \{\mathbf{s}^{(\ell)}\}_{\ell=1}^{M_q}$. Signal space expansion of the received signal at kth antenna at sequence number n is

$$\mathbf{x}_{n,k} = \sum_{i=1}^{N_T} g_{ki}\mathbf{s}_{n,i} + \mathbf{w}_{n,k}. \tag{4}$$

Notice (as a consequence of assuming nonlinear modulation) that each signal has dimensionality N_h given by the dimension of space spanned by the modulation function $h(.)$. The vector $\mathbf{w}_{n,k}$ is the signal space expansion of the Gaussian noise. It is rotationally invariant complex Gaussian random vector with zero mean and covariance matrix $\mathbf{C_w} = 2N_0\mathbf{I}$ and its PDF (probability density function) is $p_{\mathbf{w}}(\mathbf{w}) = \exp(-\|\mathbf{w}\|^2/(2N_0))/(2\pi N_0)^{N_h}$. Vectors $\mathbf{w}_{n,k}, \mathbf{w}_{n',k}$ are independent for $n \neq n'$.

The overall MIMO system model with *multidimensional* channel symbols can be now easily written with the help of *stacked* matrices as

$$\tilde{\mathbf{x}}_n = \tilde{\mathbf{G}}\tilde{\mathbf{s}}_n + \tilde{\mathbf{w}}_n. \tag{5}$$

$\tilde{\mathbf{x}}_n = [\mathbf{x}_{n,1}^T, \ldots, \mathbf{x}_{n,N_R}^T]^T$, $\tilde{\mathbf{s}}_n = [\mathbf{s}_{n,1}^T, \ldots, \mathbf{s}_{n,N_T}^T]^T$, $\tilde{\mathbf{w}}_n = [\mathbf{w}_{n,1}^T, \ldots, \mathbf{w}_{n,N_R}^T]^T$ and $\tilde{\mathbf{G}} = \mathbf{G} \otimes \mathbf{I}_{N_h}$. The symbol $\otimes$ denotes the Kronecker matrix product.

IV. CAPACITY OF FINITE ALPHABET MULTIDIMENSIONAL SIGNALS IN MIMO CHANNEL

A. Channel eigenmodes

Having the channel model in the matrix form (5), we can easily decompose the channel into the eigenmodes

$$\tilde{\mathbf{x}}_n = \tilde{\mathbf{U}}\tilde{\mathbf{D}}\tilde{\mathbf{V}}^H\tilde{\mathbf{s}}_n + \tilde{\mathbf{w}}_n \tag{6}$$

where $\tilde{\mathbf{G}} = \tilde{\mathbf{U}}\tilde{\mathbf{D}}\tilde{\mathbf{V}}^H$ is a Singular Value Decomposition (SVD) of the stacked channel matrix. One can easily verify that $\tilde{\mathbf{U}} = \mathbf{U} \otimes \mathbf{I}_{N_h}$, $\tilde{\mathbf{D}} = \mathbf{D} \otimes \mathbf{I}_{N_h}$, $\tilde{\mathbf{V}} = \mathbf{V} \otimes \mathbf{I}_{N_h}$ where $\mathbf{G} = \mathbf{U}\mathbf{D}\mathbf{V}^H$ is the SVD of channel matrix $\mathbf{G}$. The $(N_R \times N_T)$ matrix $\mathbf{D}$ is diagonal and has the first N_G nonzero diagonal elements $\alpha_1, \ldots, \alpha_{N_G}$ where $N_G = \operatorname{rank}(\mathbf{G}\mathbf{G}^H)$ is the rank of the channel. Values α_i are square roots of the eigenvalues of $\mathbf{G}\mathbf{G}^H$, $\alpha_i = \sqrt{\lambda_i} = \sqrt{\operatorname{eig}_i(\mathbf{G}\mathbf{G}^H)}$. The covariance matrix of the noise vector is $\mathbf{C}_{\tilde{\mathbf{w}}} = 2N_0\mathbf{I}$.

It is a known fact ([8]) that the unitary (a true unitary, i.e., the square matrix) transformation has unity Jacobian and therefore it does not change the entropy of *continuous-valued* random vector. Also, unitary transformation preserves zero mean Gaussian distribution with a diagonal covariance matrix. The channel eigenmodes become

$$\tilde{\mathbf{x}}'_n = \tilde{\mathbf{D}}\tilde{\mathbf{s}}'_n + \tilde{\mathbf{w}}'_n \tag{7}$$

where $\tilde{\mathbf{x}}'_n = \tilde{\mathbf{U}}^H\tilde{\mathbf{x}}_n$, $\tilde{\mathbf{s}}'_n = \tilde{\mathbf{V}}^H\tilde{\mathbf{s}}_n$, $\tilde{\mathbf{w}}'_n = \tilde{\mathbf{U}}^H\tilde{\mathbf{w}}_n$. The dimensionality of each eigenmode is N_h.

B. Informationally equivalent channel

Unitary transformation applied on *finite* alphabet symbols also preserves the mutual information and thus

$$I(\tilde{\mathbf{s}}_n;\tilde{\mathbf{x}}_n) = I(\tilde{\mathbf{s}}'_n;\tilde{\mathbf{D}}\tilde{\mathbf{s}}'_n + \tilde{\mathbf{w}}'_n) = I(\tilde{\mathbf{s}}_n;\tilde{\mathbf{D}}\tilde{\mathbf{s}}_n + \tilde{\mathbf{w}}_n). \tag{8}$$

However this is justified by the fact that the unitary transformation preserves Gram matrix and *distances* between symbols and thus the conclusions of Sec. III-B apply. This contrasts with the continuous-valued case where the justification relies on the Jacobian.

We obtain the *informationally equivalent channel* model

$$\tilde{\mathbf{y}}_n = \tilde{\mathbf{D}}\tilde{\mathbf{s}}_n + \tilde{\mathbf{w}}_n \tag{9}$$

where $\tilde{\mathbf{s}}_n$ (the same one as in (6)) is the equivalent signal space representation (Sec. III-B) of the transmitted signal and $\tilde{\mathbf{y}}_n = [\mathbf{y}_{n,1}^T, \ldots, \mathbf{y}_{n,N_R}^T]^T$ is the new virtual channel output with informationally equivalent properties as $\tilde{\mathbf{x}}_n$, $I(\tilde{\mathbf{s}}_n;\tilde{\mathbf{x}}_n) = I(\tilde{\mathbf{s}}_n;\tilde{\mathbf{y}}_n)$.

A *replacement* of $\tilde{\mathbf{s}}'_n$ by $\tilde{\mathbf{s}}_n$ in the eigenmodes is a *necessary* step that later allows the evaluation of the capacity in the random channel. *Actual codewords* are required for that evaluation, unlike from the case of continuous-valued input where only a stochastic properties of the input are needed. If we had not replace $\tilde{\mathbf{s}}'_n$ by $\tilde{\mathbf{s}}_n$, the presence of random eigenvectors would have made the average capacity evaluation more difficult. Multidimensional averaging over both *eigenvectors* and *eigenvalues* would have been needed. When we use (9) only averaging over eigenvalues is needed which *reduces* the numerical computational load substantially.

C. Symmetric capacity

In this paper, our goal is the evaluation of the *symmetric capacity* (denoted by C_*). It is defined as the mutual information between channel input and output (see Fig. 1,2) with the uniform distribution of input symbol probabilities $P_m = P_{m,i} = \Pr\{c_{n,i} = c^{(m)}\} = 1/M_c$ for all i. See [9] for a numeric iterative procedure finding the true capacity achieving distribution in a general case.

The *true* capacity (maximized over all input distributions) is equivalent to the symmetric capacity if the capacity achieving distribution is uniform. It happens in many cases, but unfortunately there is no general way of finding this. We can formulate number of *sufficient* (however generally not necessary) conditions for this situation. It can be shown [7] (based on a similar arguments as in [10, sec. 7.1.2]) that one of such special situations is the following case of *symmetric* signal set. If the set of all possible signal differences with suitable unitary transformation does not depend on k, i.e.,

$$\forall k \exists \mathbf{U} : \{\mathbf{U}(\mathbf{s}^{(m)} - \mathbf{s}^{(k)})\}_m = \{\mathbf{s}^{(m)} - \mathbf{s}^{(1)}\}_m \tag{10}$$

then the the capacity achieving probability distribution is *uniform* at the level of $\mathbf{s}$ symbols (assuming IID symbols).

In the following, we assume no channel state information at the transmitter and the perfect one at the receiver.

D. Capacity factorization per one eigenmode

An important consequence of our ability to express the finite alphabet channel with equivalent signal space representation of the waveforms directly applied to eigenmodes (see Sec. IV-B and (9)) is the possibility of the capacity factorization per one eigenmode. The total capacity *per one channel symbol* is a sum of capacities per one eigenmode. Assuming identical modulators for each antenna, it gives

$$C_{*\mathrm{MIMO}}(\boldsymbol{\alpha}) = \sum_{i=1}^{N_G} C_{*i}(\alpha_i). \tag{11}$$

The index i denoting the particular eigenmode/antenna is *dropped* for the notation simplicity in the whole subsequent treatment. The symmetric capacity per one eigenmode is obtained from the mutual information expressed in terms of the entropy

$$C_*(\alpha) = I(\mathbf{s}_n;\mathbf{y}_n)|_{P_m=\frac{1}{M_c}} = \mathcal{H}[\mathbf{y}_n] - \mathcal{H}[\mathbf{w}_n]|_{P_m=\frac{1}{M_c}} \tag{12}$$

where $\mathcal{H}[\mathbf{w}_n] = N_h \log_2(2\pi N_0 e)$. The capacity depends on particular eigenvalue α.

E. Memoryless modulator—IID channel symbols

We first assume that the discrete part of the modulator is *memoryless*, i.e., $q_n = c_n$, $M_q = M_c$ and the nth equivalent transmitted signal $\mathbf{s}_n = \mathbf{s}(c_n)$ depends exclusively on nth symbol c_n. The entropy of received signal in one eigenmode is $\mathcal{H}[\mathbf{y}_n] = -\mathrm{E}[\log_2 p_{\mathbf{y}_n}(\mathbf{y}_n)]$ where

$$p_{\mathbf{y}_n}(\mathbf{y}_n) = \sum_{m=1}^{M_q} P_m p_{\mathbf{w}}(\mathbf{y}_n - \alpha\mathbf{s}^{(m)}). \tag{13}$$

The symmetric capacity is

$$C_*(\alpha) = -\mathrm{E}[\log_2 \sum_{m=1}^{M_q} \frac{1}{M_c} p_{\mathbf{w}}(\mathbf{y}_n - \alpha\mathbf{s}^{(m)})] - \mathcal{H}[\mathbf{w}_n]. \tag{14}$$

F. Channel symbols as a Markov chain

Now we extend the previous results into the case when the modulator discrete part has *memory*. The channel symbols at one antenna (*dropping* the antenna index) depend on the data and modulator state $q_n = q(c_n, \sigma_n)$. The equivalent transmitted signal is then a function of the channel symbols $\mathbf{s}_n = \mathbf{s}(q_n)$. The sequence $\{q_n\}_n$ forms a *stationary* Markov chain (see Sec. II). The mapping $q_n \mapsto \mathbf{s}_n$ for the individual eigenmode is also assumed to be a one-to-one mapping. The equivalent signal space representation of transmitted signal $\mathbf{s}_n$ also forms a Markov chain with the same transition matrix $\mathbf{\Pi}$. The output $\mathbf{y}_n$ of the equivalent channel model forms a *hidden* Markov chain.

The capacity evaluation in the case of the output sequence with hidden Markov property is somewhat more involved. Average mutual information per one symbol can be expressed as function of *entropy rates* $\bar{I}(\mathbf{s}_n; \mathbf{y}_n) = \bar{\mathcal{H}}[\mathbf{y}_n] - \bar{\mathcal{H}}[\mathbf{w}_n]$. As a consequence of being white, the noise entropy rate is $\bar{\mathcal{H}}[\mathbf{w}_n] = \mathcal{H}[\mathbf{w}_n] = N_h \log_2(2\pi N_0 e)$. The entropy rate of the channel output can be lower and upper bounded [11]

$$\bar{\mathcal{H}}[\mathbf{y}_n] \geq \mathcal{H}[\mathbf{y}_n|\mathbf{y}_{n-1}, \ldots, \mathbf{y}_1, \mathbf{y}_0, \mathbf{s}_0], \quad (15)$$

$$\bar{\mathcal{H}}[\mathbf{y}_n] \leq \mathcal{H}[\mathbf{y}_n|\mathbf{y}_{n-1}, \ldots, \mathbf{y}_1, \mathbf{y}_0]. \quad (16)$$

The bounds approach the $\bar{\mathcal{H}}[\mathbf{y}_n]$ as n increases. However, the required dimensionality of integration when evaluating the conditional entropy also increases making this rather mathematically intractable. Therefore we use a *first order approximation*

$$\mathcal{H}[\mathbf{y}_1|\mathbf{y}_0, \mathbf{s}_0] \leq \bar{\mathcal{H}}[\mathbf{y}_n] \leq \mathcal{H}[\mathbf{y}_1|\mathbf{y}_0]. \quad (17)$$

Skipping details, we can get for these bounds $\mathcal{H}[\mathbf{y}_1|\mathbf{y}_0] = \mathcal{H}[\mathbf{y}_1]$ and

$$\mathcal{H}[\mathbf{y}_1|\mathbf{y}_0, \mathbf{s}_0] = -\int_{-\infty}^{\infty} \sum_{m=1}^{M_q} P_m \sum_{k=1}^{M_q} \Pi_{m,k} p_{\mathbf{w}}(\mathbf{y}_1 - \alpha \mathbf{s}^{(k)}) \times \log_2 \sum_{j=1}^{M_q} \Pi_{m,j} p_{\mathbf{w}}(\mathbf{y}_1 - \alpha \mathbf{s}^{(j)})\, d\mathbf{y}_1. \quad (18)$$

It is easy to see that the symmetric capacity Markov model *upper bound* (first order approximation) $C_*^{\text{MUB}}(\alpha)$ is *equivalent* to the *memoryless case* $C_*^{\text{MUB}}(\alpha) = C_*(\alpha)$ treated in the previous section. Unfortunately, this makes the bound rather loose since it completely ignores the memory. There is also, another *trivial finite alphabet upper bound* $C_*^{\text{TUB}} = \log_2 M_c$ which holds *generally* for *arbitrary* (e.g., no envelope constraint) finite alphabet modulation. A combination of these two can be used to narrow the gap

$$C_*(\alpha) \leq C_*^{\text{UB}}(\alpha) = \min\left(C_*^{\text{TUB}}, C_*^{\text{MUB}}(\alpha)\right). \quad (19)$$

The symmetric capacity *lower bound* of *one eigenmode* is

$$C_*^{\text{MLB}}(\alpha) = \mathcal{H}[\mathbf{y}_1|\mathbf{y}_0, \mathbf{s}_0] - N_h \log_2(2\pi N_0 e)|_{P_m = \frac{1}{M_c}}. \quad (20)$$

G. Mean capacity for random channel

For random IID Rayleigh channel, the channel eigenvalues obey Wishart distribution [2]. But unlike from the continuous-valued channel input, the eigenmode capacity is expressed in terms of $\alpha_i = \sqrt{\lambda_i}$. The total average ergodic symmetric capacity for IID symbols (and similarly upper/lower bounds for the Markov case) is got by the averaging

$$\bar{C}_{*\text{MIMO}} = N_G \, \mathrm{E}_\alpha[C_*(\alpha)] \quad (21)$$

where $N_G = \min(N_T, N_R)$. The averaging E_α is done with respect to the one eigenmode *marginal* distribution from the *unordered* joint one. Clearly, the PDF is $p_\alpha(\alpha) = 2\alpha p_\lambda(\alpha^2)$ where

$$p_\lambda(\lambda) = \frac{1}{a} \sum_{k=0}^{a-1} \frac{k!}{(k+b-a)!} (L_k^{b-a}(\lambda))^2 \lambda^{b-a} e^{-\lambda}. \quad (22)$$

The function $L_k^a(\cdot)$ is Laguerre polynomial, $a = \min(N_T, N_R)$, $b = \max(N_T, N_R)$. Since we used (9), no random eigenvectors were needed.

V. Example numerical results

The symmetric capacity behavior of the waveform and optionally memory constrained nonlinear modulation in MIMO channel is now demonstrated on examples. The linear modulation serves as a reference allowing to demonstrate the influence of higher dimensionality of nonlinear modulation. We chose *constant* symbol energy schemes. Examples share a common $(2, 2)$ Rayleigh MIMO channel with rank $N_G = 2$. The mean received symbol energy to noise PSD ratio per one receiver is $\gamma = \mathrm{E}[\|\tilde{\mathbf{s}}\|^2]/(2N_0)$. We show also a result with *no input alphabet constrain* [12] with the same dimensionality N_h for a comparison. Integrations in the capacity expressions were evaluated by Monte Carlo method over the marginal eigenvalue. See Fig. 3 and 4 for the mean capacity results.

MSK: MSK is a nonlinear modulation with memory that guarantees the phase continuity. The input data symbols are binary $M_c = 2$ and the modulation waveforms are $\{h^{(i)}(t)\}_{i=1}^{M_q} = \{\exp(j\frac{\pi}{2}(\sigma^{(m)} + c^{(\ell)}\frac{t}{T_S}))v(t)\}_{m,\ell}$ where $v(t) = (\mathcal{U}(t) - \mathcal{U}(t - T_S))/\sqrt{T_S}$, and $\mathcal{U}(\cdot)$ is a unit step function. The state and input symbol alphabets are $\sigma^{(m)} \in \{0, 1, 2, 3\}$, $c^{(\ell)} \in \{\pm 1\}$. There is $M_q = 8$ distinct waveforms. The state equation is $\sigma_{n+1} = (\sigma_n + c_n) \bmod 4$. The equivalent signal space representation is

$$\mathbf{S} = \begin{bmatrix} -a & aj & -aj & -a & a & -aj & aj & a \\ -b & -bj & -bj & b & b & bj & bj & -b \end{bmatrix} \quad (23)$$

where $a \cong 0.905$, $b \cong 0.426$. The dimension of the space spanned by the modulation waveforms is $N_h = 2$ per one antenna. This is a direct consequence of MSK nonlinearity. The modulator transition matrix is

$$\mathbf{\Pi} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}. \quad (24)$$

MSK class waveforms, IID symbols: This example uses the set of MSK waveforms but no memory in the discrete part of the modulator is present. The waveform is directly controlled by the input data symbols $q_n = c_n$. This also means that both alphabets have $M_c = M_q = 8$. Removing the constraint on the memory of the previous MSK case (the memory was the price we paid for controlling the continuity of the MSK phase) increased the size of input data alphabet from 2 to 8.

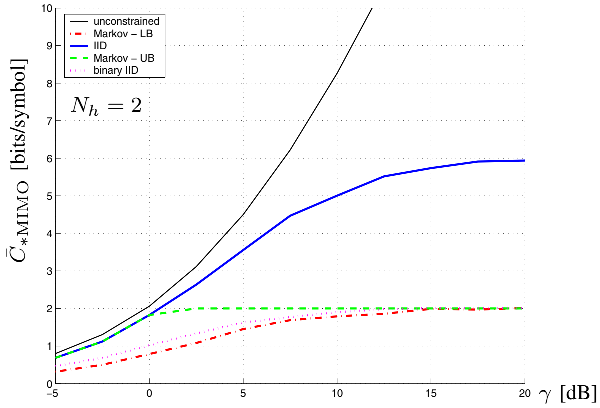

Fig. 3. MSK waveforms ($N_h = 2$) in $(2, 2)$ Rayleigh MIMO channel.

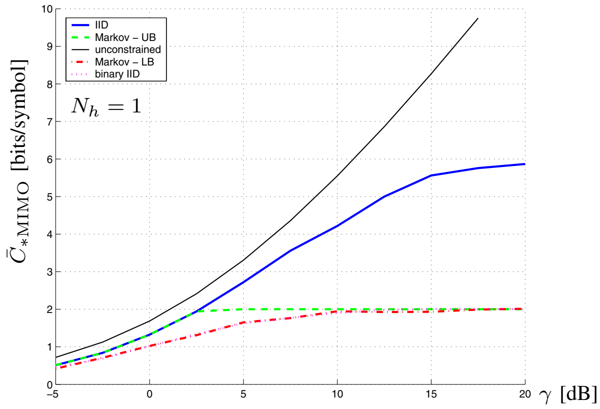

Fig. 4. 8PSK waveforms ($N_h = 1$) in $(2, 2)$ Rayleigh MIMO channel.

8PSK, IID symbols: The 8PSK modulation is a linear one serving as a reference case with the same waveform set size $M_q = 8$ and symbol energy as for MSK case but with the dimensionality $N_h = 1$. The informationally equivalent signal space representation is $\mathbf{S} = [\mathbf{s}^{(1)}, \ldots, \mathbf{s}^{(M_q)}]$ where $\mathbf{s}^{(i)} = \exp(j2\pi(i-1)/M_q)$ is a 1-dimensional vector.

8PSK with trellis code: The 8PSK waveform "π-mapper" (π phase shift for symbol pairs 1-2,3-4,5-6,7-8) is preceded by a binary to 8-ary trellis coder with the same transition matrix as in the MSK in order to investigate the influence of the dimension N_h.

Binary IID modulation: Both MSK and 8PSK with trellis code have $M_c = 2$. We included also the binary IID constellation using the best waveform pair with given symbol energy for a comparison. Such constellation is given uniquely (antipodal signals) and has dimension $N_h = 1$.

VI. Discussion of the Results and Conclusions

Factorization of the MIMO capacity averaging: The factorization, enabled by the equivalent signal space representation, in fact transfers the MIMO Rayleigh problem into the marginal root-Wishart scalar one (compare with [6], [5]).

Influence of the memory: Comparing the results with the best binary IID case, we observe that the capacity degradation caused by the presence of the modulation memory in MSK case (the price we pay for a phase continuity of MSK) is relatively very small. The alphabet finiteness is dominant.

Influence of the waveform class and dimension N_h: The IID results for MSK ($N_h = 2$) versus 8PSK ($N_h = 1$) nicely demonstrate how increased waveform dimension (for given $M_q = 8$) helps to provide better constellations.

Low signal to noise ratio: Based on the numerical results, we see that the gap between unconstrained and finite IID alphabet symmetric capacity decreases with increasing per antenna dimension N_h. The information is spread over a larger number of parallel channels (dimensions) which are *less affected by the finiteness* of the alphabet compared to case of $N_h = 1$. This favors the nonlinear modulations ($N_h > 1$) over the linear ones ($N_h = 1$) especially for low alphabet size. However, these comparisons can never be completely fair since the $N_h = 1$ case provide much lower freedom for the constellation choice given the mean symbol energy. We cannot preserve distances in the constellations when comparing the case with $N_h > 1$ and $N_h = 1$ and thus we cannot clearly separate the dependence on N_h.

High signal to noise ratio: In this case, the finite alphabet multidimensional ($N_h > 1$) waveforms cannot fully exploit the higher potential of the multidimensional unconstrained channel and the finiteness limitation becomes dominant.

Approximation order: The precision of the upper and lower bound approximation for a Markov symbols case can be improved by increasing the order of the conditional entropy paid by increased dimensionality in the integrals evaluation.

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