Nonstandard Sensitivity Analyses in Frequency and Time Domains

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ABSTRACT

The programs of PSpice type only have a possibility to compute full set of sensitivities in static domain by the sens section. In paper, an extension is described for the sensitivity analysis to be used also in frequency and time domains. Moreover, a method for determining nonstandard outputs as sensitivity of noise figure or sensitivity on a local warming up somewhere in circuit is proposed. Nonstandard sensitivity analyses are demonstrated by solving two examples—the first practical one from RFIC design area contains an improvement of noise figure accuracy, and the second special one serving as a natural test of algorithm precision in static and time domains. Derived methods have been implemented into author’s Circuit Interactive Program (C.I.A.) with necessary enhancing the efficiency of algorithms by reusing components of LU-factorizations created by frequency and time analyses.

1. FREQUENCY DOMAIN

At first, consider frequency domain. The system of parametric linear (or linearized at an operating point) equations of a circuit can be written in a matrix form

\[ A(p) \ x(p) = b(p), \]

where \( p \) is one of circuit parameters on which the sensitivities are requested. The vector of derivatives with respect to that parameter \( \partial x(p)/\partial p \) marked \( x'(p) \) can be obtained by differentiating (1)

\[ A'(p) \ x(p) + A(p) \ x'(p) = b'(p), \]

which gives a system of complex linear equations

\[ A(p) \ x'(p) = b'(p) - \frac{A(p + \Delta p) \ x(p) - b(p)}{\Delta p}, \]

if using linear approximation of the matrix derivative

\[ A'(p) \approx \frac{A(p + \Delta p) - A(p)}{\Delta p}. \]

As most of programs cannot differentiate with respect to circuit parameters in a symbolic way, that approach is usually needful. (However, topical C.I.A. edition is now able to compute matrix derivative symbolically.)

1.1. Standard sensitivity analysis

If the circuit is excited by an independent source only, the derivative \( b'(p) \) in the right side of (3) is zero and that system may be used without any modification:

\[ A(p) \ x'(p) = \frac{b(p) - A(p + \Delta p) \ x(p)}{\Delta p}. \]

1.2. Noise sensitivity analysis

The circuit is excited by \( n \) noise sources—they are not assumed to be correlated in programs of PSpice type. The sensitivity \( c'_i \) of \( i \) variable \( c_i \) is expressed by real (labeled by \( \Re \)) and imaginary (labeled by \( \Im \)) components’ parts and its derivatives

\[ c_i = \sqrt{\left( \sum_{j=1}^{n} (\Re x'_i + \Im x'_i) \right)^2 + \left( \sum_{j=1}^{n} (\Im x'_i - \Re x'_i) \right)^2}, \]

where each of the \( n \) components is obtained by solving

\[ A(p) \ jx(p) = jb(p), \]

which is formally of the same type as (1)—therefore, a complex LU-factorization of \( A \) must be executed only once for each frequency. If using analogy of (4)

\[ j b'(p) \approx \frac{j b(p + \Delta p) - j b(p)}{\Delta p}, \]

included into (3), the following sequence of systems of equations is obtained for the components’ derivatives

\[ A(p) \ jx'(p) = \frac{j b(p + \Delta p) - A(p + \Delta p) jx(p)}{\Delta p}, \]

\[ j = 1, \ldots, n; \]

however, the solutions of (9) are not difficult because the complex matrix \( A \) (stored in memory as \( L \) and \( U \) factors during the solution of (1)) is the same \( \forall j \).

1.3. Noise figure sensitivity analysis

An unconventional method for computing the noise figure \( F_n \) (or \( F_n^{\text{dB}} \) in decibels) for programs of PSpice type has been described in [1]—it evaluates formulae

\[ F_n = \frac{V_{n}^2 - V_{n, \text{Read}}^2}{A_n^2 4kT R_{\text{source}}}, \]

\[ F_n^{\text{dB}} = 10 \log F_n. \]
for all selected frequencies; \( V_n, A_V, k, T, \) and \( R_{source} \) are the total output noise voltage, voltage gain, Boltzmann constant, absolute temperature, and interior resistance of input source, respectively—the \( V_{n,R_{load}}^2 \) is a portion of \( V_n^2 \) caused by load resistance (the load is not a part of amplifier and therefore its noise must be subtracted). Note that the \( V_n, V_{n,R_{load}}, \) and \( A_V \) are normal outputs of PSpice frequency & noise analyses.

The method [1] can be improved in the two ways:

- The circuit must be matched in advance because the Friis noise figure definition assumes available noise powers and available gains.
- The subtraction of \( V_{n,R_{load}}^2 \) in (10) is performed manually for each frequency [1]. However, if the load is created artificially as a controlled current source then the load’s noise is not generated.

By differentiating (10), the requested sensitivity of \( F_n^{dB} \) on \( p \) marked \( F_n^{dB}' \) is obtained as a simple formula (using the C.I.A., \( V_{n,R_{load}} \) is defined—see example below)

\[
F_n^{dB}' = \frac{20}{\ln 10} \left( \frac{V_n'}{V_n} - \frac{A_V'}{A_V} \right),
\]

where the sensitivities \( V_n' \) and \( A_V' \) are computed by C.I.A. in the way defined in 1.2 and 1.1, respectively.

The output (11) is exceptional—most of programs cannot determine \( V_n' \) (and consequently \( F_n^{dB} \)) too.

2. TIME DOMAIN

Consider time domain now. Before the time domain sensitivity analysis can be defined, it is necessary to briefly introduce the C.I.A. algorithm for solving a system of nonlinear algebraic-differential equations.

2.1. Algorithm for solving a system of circuit algebraic-differential equations

The system of algebraic-differential equations of a circuit is generally in an implicit form

\[
f(x(t), \dot{x}(t), t) = 0.
\]

Let assume the first \( n \) steps of a numerical integration of (12) have finished. Let mark \( x(t_n) \) by \( x_n \) and define backward scaled differences by recursive formulae

\[
\delta^{(0)} x_n = x_n,
\]

\[
\delta^{(i)} x_n = \delta^{(i-1)} x_n - a_n^{(i-1)} \delta^{(i-1)} x_{n-1}, \quad i = 1, \ldots, k_n + 2,
\]

where \( k_n \) is order of polynomial interpolation to be used in the running integration step and

\[
a_n^{(0)} = 1,
\]

\[
a_n^{(i)} = a_n^{(i-1)} \frac{t_n - t_{n-i}}{t_{n-1} - t_{n-1-i}}, \quad i = 1, \ldots, k_n + 1.
\]

Prediction of values at next chosen time \( t_{n+1} \) \( x_n^{(0)} \) is determined by the explicit extrapolation using the backward scaled differences (13)

\[
x_{n+1}^{(0)} = \sum_{i=0}^{k_n} a_n^{(i)} \delta^{(i)} x_n.
\]

Correction of the values at \( t_{n+1} \) is determined using modified Newton iterations

\[
\left[ \frac{\partial f}{\partial x} \right]_{n+1} + \gamma_{n+1} \left( \frac{\partial f}{\partial x} \right)_{n+1} \Delta x_{n+1} = -f_n^{(j+1)},
\]

\[
\gamma_{n+1} = \sum_{i=1}^{k_n} \frac{1}{t_{n+1} - t_{n+1-i}},
\]

therefore, the factor \( \gamma_{n+1} \) in (16) can be expressed

\[
\gamma_{n+1} = \sum_{i=1}^{k_n} \frac{1}{t_{n+1} - t_{n+1-i}}
\]

which gives a simple formula \( \gamma_{n+1} = 1/(t_{n+1} - t_n) = 1/\Delta t \) if first order (Euler) method used, for example.

After resolving each of the linear systems (16), the vectors \( x_n^{(j+1)} \) and \( \dot{x}_n^{(j+1)} \) are updated

\[
\dot{x}_{n+1}^{(j+1)} = x_{n+1}^{(j+1)} + \Delta x_{n+1},
\]

\[
\dot{x}_{n+1}^{(j+1)} = x_{n+1}^{(j+1)} + \gamma_{n+1} \Delta x_{n+1}.
\]

The C.I.A. algorithm is a little more flexible than Gear’s method implemented in PSpice with respect to quicker step and order alterations.

2.2. Time domain sensitivity analysis

With respect to the above notation, a system of parametric algebraic-differential equations of a circuit is in a form

\[
f(x(t, p), \dot{x}(t, p), t, p) = 0.
\]

The vector of derivatives of circuit variables with respect to a circuit parameter \( p \) \( \partial x(t, p)/\partial p \) marked \( \dot{x}'(t, p) \) can be obtained by total differentiating (21)

\[
\frac{\partial f}{\partial x}(t, p) \dot{x}'(t, p) + \frac{\partial f}{\partial x}(t, p) \dot{x}'(t, p) + f'(t, p) = 0;
\]

for example, if the first order implicit method (note that the C.I.A. algorithm uses methods up to sixth order)

\[
\dot{x}'(t, p) \approx \frac{\dot{x}'(t, p) - \dot{x}'(t - \Delta t, p)}{\Delta t}
\]
is used as approximation, we obtain recursive formula
\[
\left[ \frac{\partial f}{\partial x}(t,p) + \frac{1}{\Delta t} \frac{\partial f}{\partial \tau}(t,p) \right] x'(t,p) = -f'(t,p) + \frac{1}{\Delta t} \frac{\partial f}{\partial \tau}(t,p) x'(t-\Delta t,p), \tag{24}
\]
where the last member of (24) represents a “memory” of the sensitivities. Emphasize that the Jacobian in (24) is the same as that in (16)—therefore, a laborious LU-factorization must be executed once for each time. See also [2] as another unusual way in such analyses.

### 2.3. Operating point sensitivity analysis

For determining initial parametric derivative \( x'(\Delta t, p) \), a static parametric derivative \( x'(0, p) = x_0(p) \) must be determined in advance—see (24). That vector can be obtained by differentiating a static version of (21)
\[
f_0(x_0(p), p) = 0, \tag{25}
\]
which gives a simpler system of linear equations
\[
\frac{\partial f_0}{\partial x_0}(p) x'_0(p) = -f'_0(p) \tag{26}
\]
to be solved to obtain that starting vector for (24).

### 3. EXAMPLES

#### 3.1. Frequency domain

Consider a monolithic microwave integrated circuit (MMIC) in Fig. 1 for which the noise figure sensitivity analysis is performed. The C.I.A. match of the circuit announced above has given values of source interior \( R_{\text{source}} \) and load \( R_{\text{load}} \) resistances 20 \( \Omega \) and 300 \( \Omega \), respectively. (Here, only the impedance matching at 1.5 GHz used—a more exact way is suggested in [3]; however, the values 20/300 \( \Omega \) are better substantially than 50/50 \( \Omega \) used in [1].) Noiseless load resistor is defined by functional C.I.A. input language as current source \((V_7 - V_6)/300 \Omega \) (for controlled sources, PSpice and C.I.A. do not generate any noise contribution).

All the results are summarised in Fig. 2. The noise figure is about 3.18 dB at 1.5 GHz for the 50/50 \( \Omega \) unmatched circuit (3.15 dB declared in [1]). For the 20/300 \( \Omega \) matched circuit, the noise figure is about 2.22 dB at 1.5 GHz. All the scaled (per one \%) sensitivities of \( F_n^{\text{dB}} \) are drawn by solid lines. At 1.5 GHz, \((\partial F_n^{\text{dB}}/\partial \tau)(\tau_F/100)\) is dominant—about 0.0122 dB. The transit time \( \tau_F \) of QRF6E20 was 28 ps—for recent technologies, there is no problem to use faster transistors with \( \tau_F = 21 \) ps (~25 \%), e.g. The \( F_n^{\text{dB}} \) for such updated circuit is now about 1.95 dB at 1.5 GHz which confirms a noise figure estimation from that sensitivity (2.22 – 25 × 0.0122 = 1.915—to be precise, a new matching has to be done).

#### 3.2. Time domain

Consider a power operational amplifier in Fig. 3 for which both static and dynamic sensitivity analyses will be demonstrated. The amplifier has input transistors symmetrically connected as standard differential pair. Therefore, a sensitivity of output voltage \( v_1 \) on local warming up \( \Delta T_1 \) of B1 is to be complementary to a sensitivity of \( v_1 \) on local warming up \( \Delta T_2 \) of B2. The values of those nonstandard sensitivities will be monitored as an appropriate test of the algorithm.

#### 3.2.1. Operating point sensitivities

The operating point sensitivities obtained by solving static system (26) are zero-symmetrical as expected
\[
\frac{\partial v_1}{\partial (\Delta T_1)} = +1.925071 \times 10^{-3} \text{ V/K}, \quad \frac{\partial v_1}{\partial (\Delta T_2)} = -1.929745 \times 10^{-3} \text{ V/K}.
\]
Fig. 3. Power operational amplifier for testing the static and dynamic sensitivity analyses.

In other words, if the two transistors have the same warming up (which is natural) by 10 K, the voltage $v_1$ changes approximately by $-50 \mu V$—the precise symmetry is very important because the output voltage is only about $1.3 \text{ mV}$ at amplifier’s operating point.

3.2.2. Time domain sensitivities

The results of solving the recursive system (24) for the first period with starting vector $x'_0(p)$ are drawn in Fig. 4. The best absolute symmetry occurred—as also expected—when the input source and output voltage are close to zero values. For other voltages, relative symmetry with respect to $v_1$ is also sufficient.

The dynamic sensitivity analysis itself is essential for identification of dynamic models by optimization.

4. CONCLUSION

Enhancements of sensitivity analysis in frequency and time domains are presented. The way for computing nonstandard important outputs as sensitivity of noise figure or sensitivity on local warming up somewhere in circuit is described. Accuracy of the noise figure and its sensitivities is increased by circuit matching.

5. REFERENCES

