Non-Cooperative Compute-and-Forward Strategies in Gaussian Multi-Source Multi-Relay Networks

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Abstract—Compute-and-forward is an emerging technique to increase throughput in interference networks. It allows the relays to decode an equation of codewords instead of each single codeword. Choosing the appropriate equation plays a crucial role on the performance. While there are several algorithms that solve this optimization problem, they all perform a local optimization. In large relay networks this can result in linear dependent equations at the destination and a large outage probability. In this paper we propose new non-cooperative strategies on how to choose the desired equations and enforce linear independent equations. This allows to solve the optimization problem in a localized and distributed manner. Further we compare the achievable sum-rate of those strategies and show that correlated signals have a huge impact on the performance.

1. INTRODUCTION

Network Coding has been a very promising topic since introduced by Ahlswede et al. in [1]. The concept of network coding in wired networks is very well investigated and first implementations for industrial usage are emerging [2], [3]. Wireless networks on the other hand are still subject of intensive research. The properties of the wireless channel allow network coding on different layers. The superposition property can be exploited by the relays and enables Wireless Physical Layer Network Coding (WPLNC). Since the introduction of physical layer network coding [4], different schemes have been developed, e.g., [5]–[8]. A framework that has drawn a lot of attention is compute-and-forward (CF) [9]. It uses lattice codes and exploits their structure to allow the decoding of linear equations of codewords without decoding the codewords themselves. This enables the relaying nodes to decode the equations at higher rates than it would be possible by decoding the individual messages, e.g., by decode-and-forward. A further important advantage compared to e.g., amplify-and-forward is the possibility to avoid noise accumulation, which reduces the performance in large networks. Also this has a lot of advantages and high rates are achievable, this comes with the need for network diversity. The final destination needs to collect enough independent equations to jointly decode the transmitted data.

The performance of CF highly depends on the alignment of the channel coefficients with the coefficients of the decoded equation. Several algorithms have been proposed to solve for the optimal equation in terms of achievable computation rate [10]–[12]. While most of these algorithms perform a local optimization, they ignore the network structure and the possible linear dependence of the equations at the final destination. This will result in outages because the destination is not able to decode the messages if it has not enough linear independent equations. This is a very serious problem in real applications, especially in large networks with several hops because re-transmissions from all sources to the destination are not applicable. The delay might be extremely large and the signaling will pollute the network. This problem can be overcome by allowing cooperation between nodes [12]. In realistic setups however, this might not be applicable due to the large signaling overhead to allow this kind of cooperation. Further, it is not very robust against channel state changes. Therefore the question arises if it is possible to exploit the network structure such that no cooperation between nodes is needed except for a network initialization phase.

In this paper we introduce some non-cooperative schemes which enforce linear independence of the decoded equations at the destination. We compare the performance of these schemes in multi-source multi-relay networks and show that correlation between channel coefficients plays a crucial role. We find that correlation can decrease the achievable sum-rate of CF by 1.5 bit/cu whereas our proposed schemes are robust against correlation and can achieve twice the sum-rate. The paper is organized as follows. We define the system model in Sec. II and introduce the different relaying schemes in Sec. III. We discuss the performance of these schemes with the help of simulation results in Sec. IV and conclude this work in Sec. V.

A. Notation

Let \( \log^+(x) \triangleq \max\{0, \log(x)\} \) and \( \mathbb{F}_p \) a finite field of size \( p \), where \( p \) is a prime. We denote by \( x' \) the transpose of vector \( x \) and by \( e_i \) the unit vector with a one at position \( i \) and zeros elsewhere.

In the following we recall some lattice definitions that are used throughout the paper. For further details on lattice codes see [6], [13]–[15]. An \( n \)-dimensional lattice \( \Lambda \subseteq \mathbb{R}^n \) is a group under addition with generator matrix \( G \in \mathbb{R}^{n \times n} \), i.e., \( \Lambda = \{GC : c \in \mathbb{Z}^n\} \). A lattice quantizer is a mapping \( Q_{\Lambda} : \mathbb{R}^n \to \Lambda \) that maps a point \( x \) to the nearest lattice point in Euclidean distance, i.e., \( Q_{\Lambda}(x) = \arg \min_{\lambda \in \Lambda} \|x - \lambda\| \). Let the modulo operation with respect to the lattice \( \Lambda \) be defined as \( [x] \mod \Lambda = x - Q_{\Lambda}(x) \). We call \( V = \{x : Q_{\Lambda}(x) = 0\} \) the fundamental Voronoi region of the lattice \( \Lambda \) and denote by \( \text{Vol}(V) \) the volume of \( V \).

Two lattices \( \Lambda_C \) and \( \Lambda_F \) are called nested, if \( \Lambda_C \subseteq \Lambda_F \).
We call $\Lambda_C$ the coarse lattice and $\Lambda_F$ the fine lattice. A nested lattice code $\mathcal{L}$ is formed by taking all of the points of the fine lattice $\Lambda_F$ in the fundamental Voronoi region $\mathcal{V}_C$ of the coarse lattice $\Lambda_C$, i.e., $\mathcal{L} = \Lambda_F \cap \mathcal{V}_C$.

II. System Model

We investigate an $L \times M$ relay network consisting of $L$ source nodes, $M$ relay nodes and one destination node as depicted in Fig. 1. Each source node $\ell$ has a message $w_\ell$ that has to be transmitted to the destination node. Because there are no direct links between the source nodes and the destination, the transmission has to use the help of several relays. For the ease of simplicity we assume that all $M$ relays are used. The relays can apply several relaying strategies which are explained in detail in Sec. III. The used relaying strategy is common to all relays and a system design parameter. We will investigate which relaying strategy is best to use for certain system parameters.

Remark 1. We want to stress that we assume no cooperation between the relays because this would drastically increase the complexity in realistic scenarios. The cost for network providers in terms of infrastructure complexity is much less when the nodes are independent. For example we have less signaling overhead. Further, node independence increases the robustness of the network against topology changes.

We will focus our work on the communication between source and relay nodes and the decoding at the relay nodes. There might be other layers or networks after the relays which are beyond the scope of the investigation. Therefore, we assume that the channels between the relays and the destination are bit-pipes with large enough capacity. They are error-free and do not interfere with each other. This might be achieved by frequency-division multiple-access (FDMA) or similar techniques.

A. General Definitions

Definition 2 (Messages). Each source node $\ell$ chooses a length-$k$ message vector $w_\ell \in \mathbb{F}_p^k$ i.i.d. from a uniform distribution over the index set $\{1, 2, \ldots, 2^{\lfloor nR_\ell \rfloor}\}$. Because we will build linear combinations of these messages, we zero-pad them to a common length $k = \max_{\ell} k_\ell$.

Definition 3 (Encoders). Each source node is equipped with an encoder,

$$E_\ell : \mathbb{F}_p^k \rightarrow \Lambda_F \cap \mathcal{V}_C,$$

that maps length-$k$ messages over the finite field to length-$n$ real-valued codewords of a lattice code $\mathcal{L} = \Lambda_F \cap \mathcal{V}_C$, i.e., $x_\ell = E_\ell(w_\ell)$. Each codeword is subject to the power constraint $\|x_\ell\|^2 \leq nP$. Therefore $\Lambda_C$ is chosen such that the second moment of $\Lambda_C$ equals $P$.

Definition 4 (Channel Model). Each relay receives a noisy linear combination of the transmitted signals through the channel, i.e.,

$$y_m = \sum_{\ell=1}^{L} h_{m\ell} x_\ell + z_m,$$

where $h_{m\ell} \in \mathbb{R}$ are the channel coefficients and $z_m$ is i.i.d. Gaussian noise, i.e., $z_m \sim \mathcal{N}(0, I_n)$. Let $h_m = (h_{m1}, \ldots, h_{mL})^T$ denote the vector of channel coefficients to relay $m$ and let $H = \{h_{m\ell}\}$ denote the entire channel matrix. With this notation the $m$-th row in $H$ is $h_m$.

Definition 5 (Desired Equations). The goal of each relay is to reliably recover a linear combination of lattice codewords

$$v_m = \left[ \sum_{\ell=0}^{L} a_{m\ell} x_\ell \right] \mod \Lambda_C.$$

The desired equation is represented by a coefficient vector $a_m = (a_{m1}, \ldots, a_{mL})^T$.

Definition 6 (Message Rate). The message rate of each source node is the length of its message normalized by the number of channel uses,

$$R_\ell = \frac{k_\ell}{n} \log_2 p.$$

Definition 7 (Computation Rate). The computation rate $\mathcal{R}(h_{m}, a_m)$ is achievable if for any $\varepsilon > 0$ and $n$ large enough, there exist encoders and decoders, such that all relays can recover their desired equations with average probability of error $\varepsilon$ so long as the underlying message rates $R_\ell$ satisfy

$$\forall \ell \in \{1, \ldots, L\} : R_\ell < \min_{m, a_m \neq 0} \mathcal{R}(h_m, a_m).$$

B. Decoding at the Destination

The destination node receives $M$ linear combinations of the lattice codewords from the relay nodes, i.e.,

$$y_D = A \cdot x,$$

where $x = (x_1, \ldots, x_L)^T$ is the vector of transmitted lattice codewords from all source nodes and $A = (a_{ij})$ with $i \in \{1, \ldots, M\}$ and $j \in \{1, \ldots, L\}$ is the coefficient matrix of the linear combinations. Please note that the $m$-th row in $A$ is equal to the coefficient vector $a_m$ of the linear combination decoded at relay $m$. The destination can solve the system of linear equations in Equation (6) via matrix inversion if the coefficient matrix $A$ has full rank, i.e., $\text{rank}(A) \geq m$. Otherwise an outage occurs.

Definition 8 (Outage Probability). The outage probability $P_{\text{out}}$ is the probability that the destination node is not able to decode each single codeword $x_\ell$ transmitted by source node $\ell$, $\ell \in \{1, \ldots, L\}$.

Definition 9 (Achievable Sum-rate). The achievable sum-rate is defined as the goodput of all source nodes to the destination,

$$R_{\text{sum}} = (1 - P_{\text{out}}) \sum_{\ell=1}^{L} R_\ell,$$

where $P_{\text{out}}$ is the outage probability.
where relay \( \alpha_m \) can be chosen such that linear independence at the destination is guaranteed. We get the following optimization problem
\[
\beta^* = \arg\max_{\beta_m \in \mathbb{Z}^m} \mathcal{R}(h_m, \beta_m).
\]
\[\beta^* \in \mathcal{R}(h_m, \beta_m).\]  
In a large network with multiple hops we get outages due to linear dependent equations at the destination. Even for a single hop with multiple relays the outage probability is very high (see Fig. 2). This motivates us to modify the optimization problem of finding the best equation such that the network structure is exploited. Therefore we introduce some modified CF schemes which differ in the way \( \alpha_m \) is chosen.

A. Classical Compute-and-Forward

With the term classical CF we refer to the unaltered scheme introduced in [9]. In the classical CF scheme each relay solves the following optimization problem individually to maximize the achievable computation rate,
\[
\mathcal{R}(h_m, \alpha_m) = \frac{1}{2} \log_2 \left( \frac{1}{\alpha_m \cdot G_{\text{CF}} \cdot \alpha_m} \right),
\]
where
\[
G_{\text{CF}} = I_{L \times L} - \frac{P}{1 + P\|h_m\|^2} \cdot h_m h_m^\dagger.
\]

CF is able to achieve a high computation rate due to the fact that it decodes a linear combination of messages and not each single message like for example decode-and-forward. However, this does not take the network structure into account. In large networks with multiple hops we get outages due to linear dependent equations at the destination. Even for a single hop with multiple relays the outage probability is very high (see Fig. 2). This motivates us to modify the optimization problem of finding the best equation such that the network structure is exploited. Therefore we introduce some modified CF schemes which differ in the way \( \alpha_m \) is chosen.

B. Single User Decoding

By single user decoding we refer to a special case of CF, where relay \( m \) decodes the linear combination \( \alpha_m = \epsilon_m \). This means each relay decodes only a single codeword and treats all other signals as noise. Because each relay decodes a different codeword, it is guaranteed that the coefficient matrix \( A \) at the destination node has full rank. Therefore we do not have outages, i.e., \( P_{\text{out}} = 0 \). However, this comes at the cost of achievable computation rate because we choose a suboptimal coefficient vector in Equation (10). The achievable computation rate at relay \( m \) is given by
\[
R_{\text{CF},m} = \mathcal{R}(h_m, \epsilon_m).
\]

C. Subspace Compute-and-Forward

In the following we introduce subspace CF. We reduce the feasible set of possible coefficient vectors in the optimization problem in Equation (10) for each relay such that the subsets are linearly independent. We get the following optimization problem at relay \( m \)
\[
\mathcal{P}_{\text{SCF},m} = \max_{\alpha_m \in \mathcal{S}_m} \mathcal{R}(h_m, \alpha_m),
\]
where \( \mathcal{S}_m \subset \mathbb{Z}^L \). For the construction of the subsets we restrict ourselves to the case where \( L = M \), i.e., the number of relay nodes equals the number of source nodes. It is obvious that in the case of \( M > L \) we get \( M - L \) linear dependent equations at the destination which are not necessary for decoding. This becomes a problem of relay selection and is beyond the scope of this paper.

Let \( B \in \mathbb{Z}^{L \times L} \) be a basis matrix for \( \mathbb{Z}^L \). We define \( B_m \) to be a sliced version of \( B \) containing only the first \( m \) rows of \( B \). The subset \( \mathcal{S}_m \) is then constructed by
\[
\mathcal{S}_m = \{ B_m \cdot \beta_m : \beta_m \in \mathbb{Z}^m, \beta_{mm} \neq 0 \}. \]

The achievable computation rate at relay \( m \) is the solution to the following optimization problem
\[
R_{\text{SCF},m} = \max_{\beta_m \in \mathcal{S}_m, \beta_{mm} \neq 0} \mathcal{R}(h_m, B_m \cdot \beta_m),
\]
which can be written as
\[
R_{\text{SCF},m} = \max_{\beta_m \in \mathcal{S}_m, \beta_{mm} \neq 0} \frac{1}{2} \log_2 \left( \frac{1}{\beta_m \cdot G_{\text{SCF}} \cdot \beta_m} \right),
\]
where
\[
G_{\text{SCF}} = B_m B_m^\dagger - \frac{P}{1 + P\|h_m\|^2} \cdot B_m h_m h_m^\dagger B_m^\dagger.
\]

Remark 11. Please note that the optimization problem has the same structure as the one for classical CF (see Equation (10)). If a sorted channel vector \( h_m \) is assumed, then the constraint sets are also equivalent (see [10] for details). Therefore we can use the same algorithms to solve this optimization problem.

D. Hierarchical Compute-and-Forward

Hierarchical CF has been introduced independently and from different points of view in [16], [17]. This scheme fixes the equations that are decoded at each relay independently from the channel realizations. Therefore the equation coefficients can be chosen such that linear independence at
the destination is guaranteed. This comes at the prize of a lower performance because the channel coefficients and the equation coefficients are not well aligned. The key technique used to improve the performance is interference cancellation. The decoder at relay $m$ decodes an auxiliary equation $\tilde{a}_m$ and subtract this from the received signal to create a new virtual channel $\tilde{h}_m = h_m - \eta \tilde{a}_m$.

The auxiliary equation has to be chosen such that the decoding performance of the desired equation $a_m$ is increased for this virtual channel. The coefficient $\eta \in \mathbb{R}$ has to be chosen such that the mean squared error between the virtual channel $\tilde{h}_m$ and the desired coefficients $a_m$ is minimized. To obtain the optimal auxiliary equation one has to solve the following optimization problem at relay $m$,

$$ R_{\text{CF},m} = \max_{\tilde{a}_m \in A_m} \min \{ R(h_m, \tilde{a}_m), R(\tilde{h}_m, a_m) \}, $$

where $A_m = \{ \tilde{a}_m : R(h_m, \tilde{a}_m) > R(\tilde{h}_m, a_m) \}$ is the set of all auxiliary equations which improve the performance. So far the optimization problem in Equation (18) lacks an efficient algorithm to solve this problem. It is basically a combinatorial problem which can be solved by an exhaustive search with high complexity.

**Remark 12.** Please note that hierarchical CF can be combined with single user decoding as well as subspace CF to improve the performance of the respective scheme. However, one has to keep in mind that this will increase the computational complexity for solving the respective optimization problems drastically and might not be worth the performance increase.

### IV. Simulation Results

In this section we provide simulation results showing the performance of the relaying schemes introduced above. We use the algorithm in [10] to obtain the coefficients for classical and subspace CF. Further we use an exhaustive search for hierarchical CF. Please note that we measure the performance solely in terms of achievable sum-rate. Some practical concerns about signaling overhead and delay are already raised in Sec. I and Sec. III. In addition to the non-cooperative schemes we also plot the achievable sum-rate for CF with cooperation between the relays as an upper bound. For that we use the algorithm provided in [12].

#### A. No Correlation

In the following we compare the achievable sum-rate of the different schemes in a $3 \times 3$ relay network, where the channel coefficients are i.i.d. according to a Gaussian distribution, i.e., $h_{m \ell} \sim \mathcal{N}(0,1)$. As one can see in Fig. 3 the classical non-cooperative CF scheme achieves approximately half a bit less sum-rate than the cooperative scheme. The subspace CF strategy performs well for low SNR but losses its performance benefit in the high SNR regime.

The performance of hierarchical CF highly depends on the chosen desired equations as one can see in Fig. 3 where we plotted the achievable sum-rate for two different desired coefficient matrices, i.e., $A = I_{3\times3}$ and

$$ A = \begin{pmatrix} 1 & 0 & 1 \ 0 & 1 & 1 \ 1 & 0 & 1 \end{pmatrix}. $$

A good choice for the desired coefficient matrix is $A = I$. One might think that this should result at least in the same achievable sum-rate as single user decoding. Unfortunately this is not true as one can see from Equation (5). The achievable computation rate of relay $m$ only constraints the rate of the source nodes whose codewords are decoded with a non-zero coefficient. If relay $m$ decodes with $a_m = e_i$ it would only affect the achievable rate of source node $i$. If relay $m$ decodes additionally an auxiliary equation with more non-zero entries, the computation rate of that auxiliary equation effects more relays. Because each source node is constrained by the minimum achievable computation rate of all relays, it is possible that the sum-rate is reduced by using an auxiliary equation at a relay although the individual computation rate at the relay benefits from using an auxiliary equation. This is one side-effect of having no cooperation between the relays.

**Remark 13.** The optimal desired coefficient matrix $A$ for hierarchical CF for a given channel distribution is still an open question and will be subject of future research.

#### B. Correlation

In this section we want to show the influence of correlation on the achievable sum-rate. Therefore we model the channel between source nodes and relay nodes by the Kronecker model,

$$ H = R_{12}^{1/2} \cdot W \cdot R_{12}^{1/2}, $$

where $W \sim \mathcal{N}(0, I_{M \times L})$ and $R_{12}$ are the receive and transmit correlation matrices, respectively. For the simulations we assume a $3 \times 3$ relay network with different correlation scenarios:

- strong correlation between two sources and two relays:

  $$ R_{12} = \begin{pmatrix} 1.0 & 0.9 & 0.1 \\ 0.9 & 1.0 & 0.1 \\ 0.1 & 0.1 & 1.0 \end{pmatrix}, \quad R_{RT} = \begin{pmatrix} 1.0 & 0.9 & 0.1 \\ 0.9 & 1.0 & 0.1 \\ 0.1 & 0.1 & 1.0 \end{pmatrix} $$

- weak correlation between two sources and strong correlation between two relays:

  $$ R_{12} = \begin{pmatrix} 1.0 & 0.9 & 0.1 \\ 0.9 & 1.0 & 0.1 \\ 0.1 & 0.1 & 1.0 \end{pmatrix}, \quad R_{RT} = \begin{pmatrix} 1.0 & 0.1 & 0.1 \\ 0.1 & 1.0 & 0.1 \\ 0.1 & 0.1 & 1.0 \end{pmatrix} $$

As one can see in Fig. 4 the correlation in the network plays an important role. Strong correlation even between only two source nodes or relay nodes significantly decreases the achievable sum-rate of the classical CF scheme (compare to Fig. 3 without correlation). This is due to the fact that a strong correlation increases the probability that the relay nodes will decode a linear dependent equation. Subspace CF and single user decoding however are robust against correlation and do not show a decrease in performance. In fact subspace CF shows a superior performance for high correlation.
In networks with several hops, where re-transmissions from source to destination are not applicable, one should use subspace CF. The decrease of achievable sum-rate compared to classical CF is approximately 1 bit/cu for an SNR of 10 dB.

- Hierarchical CF can increase the performance of subspace CF as well as single user decoding in the high SNR regime but has high computational complexity.
- Correlation plays an important role on the performance of classical CF and can reduce the achievable sum-rate to 0 bit/cu. Subspace CF, hierarchical CF as well as single user decoding are robust against correlation.
- If a high correlation between several nodes in the network occurs, subspace CF shows a superior performance.

As one can see subspace CF looks like the winning strategy. It has no increase in computational complexity compared to classical CF. It is robust against correlation and has a good performance in almost all scenarios.

**V. CONCLUSION**

In this paper we analyzed the performance of CF in multi-source multi-relay networks. Because cooperation is an obstacle in large networks we focused on non-cooperative versions of CF and proposed new strategies for choosing the desired equations at the relay nodes. We show that the choice of these equations is essential for the performance of the complete network and analyzed the impact of correlated signals in the network on the achievable sum-rate. The results can be summarized as follows:

- Classical CF
- Cooperative CF
- Hierarchical CF
- Subspace CF

For a better impression on the dependency between achievable sum-rate and correlation, we plot one over the other in Fig. 5. We consider two scenarios:

- If a high correlation between several nodes in the network occurs, subspace CF shows a superior performance.
- If a high correlation between several nodes in the network occurs, subspace CF shows a superior performance.

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