Hierarchical Alphabet and Parametric Channel Constrained Capacity Regions for HDF Strategy in Parametric Wireless 2-WRC

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***Abstract*—The paper addresses wireless the 2-Way Relay Channel (2-WRC) system with a Hierarchical Decode and Forward strategy. This strategy uses a Hierarchical eXclusive Code (HXC) that allows full decoding of the hierarchical symbols at the relay. The HXC represents two data sources only through the exclusive law and requires side information on the complementary data at the destination (which naturally holds for the 2-WRC). The HDF strategy has the advantage over classical MAC stage relaying with joint decoding that its rate region extends beyond the classical MAC region. We evaluate the hierarchical MAC capacity regions for various alphabets, constellation point indexing and various channel parametrization and compare that to the alphabet limited and unconstrained cut-set bounds.**

I. Introduction

A. Background And Related Work

Multi-node and multi-source wireless communication scenarios are currently under intensive investigation in the research community. Generally, these can be seen as similar to the Network Coding (NC) paradigm [1]. NC operates with a discrete (typical binary) alphabet over loss-less discrete channels. It is in fact an operation on data rather than on the channel codewords. NC has great potential in substantially increasing the throughput of complicated communication networks. An extension of these principles into the wireless (signal space) domain is however non-trivial. Some attempts have been carried out using a simple concatenation of NC and a single-link physical layer modulation and coding technique. This has number of drawbacks and only limited optimality. An optimal solution is a direct signal space domain code synthesis.

An NC based approach in the signal space channel is called the Physical Network Coding (PNC) or the Network Coded Modulation (NCM). It provides performance advantages over the standard technique based on Amplify & Forward or Joint Decode & Forward paradigms. The main advantage of the NCM is in the fact that the relay decodes only *hierarchical* symbols (codewords) which represents jointly both sources on condition that the complementary side information is provided. The consequence is a throughput increase of the MAC phase which is a limiting factor of the overall system. We call this a Hierarchical Decode and Forward (HDF) strategy. The simplest realization of the HDF strategy is through the modulo decoding mapping at the relay (authors of [2] call that Modulo Decoding). This forms the minimal cardinality mapping. However, other mappings with extended cardinality are possible (see Section III-A and [3]).

Limited code design and capacity region results are available for the simplest possible scenario of the parametric 2-Way Relay Channel (2-WRC), see [4], [5], [6], [7]. A specific approach using a lattice code construction is taken by [8] and [2]. The authors of [8] and [2] provide lattice based code construction using principles of [9] but do not investigate the impact of the channel parametrization which is our major goal in this paper.

B. Goals of this Paper and Contributions

We present a relaying strategy for the 2WRC based on hierarchical exclusive code relay processing. Hierarchical relay processing handles hierarchical data symbols which uniquely represent the individual data from source A and B only given the side-information on the complementary data stream. This strategy is known to have an advantage in having higher and rectangular achievable rate region in comparison with the classical MAC channel. In [3] the present authors introduce a *layered* approach to the 2-WRC, based on a *hierarchical exclusive alphabet*.

The paper exploits this approach, providing the following results and contributions.

1) We analyze the alphabet limited *achievable rate region* for the hierarchical MAC stage of the relaying and compare it to the alphabet limited and unconstrained cut-set bound capacity limits.
2) We analyze the impact of the channel *parametrization* on the above stated capacities.

II. System Model and Definitions

We consider a wireless 2-Way Relay Channel (2-WRC) system (Fig. 1) which has 3 physically separated nodes (source

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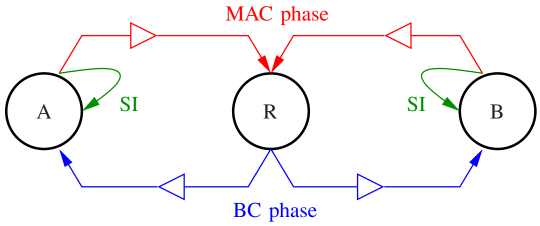

Figure 1. 2WRC with SI.

S, relay R, destination D) supporting two way communication through a common shared relay R. The source for data A is co-located with the destination for data B and vice-versa. The transmitted data (signal) of the source serves as Side Information (SI) for the destination of the reverse link. The system is wireless and all transmitted and received symbols are signal space symbols. The channel model is a linear frequency flat with Additive White Gaussian Noise (AWGN). The signal transmission is half-duplex (the co-located transmitter (Tx) and receiver (Rx) are not allowed to operate simultaneously). The operation is split into a Multiple Access (MAC) and a Broadcast (BC) phase. The system is symmetric in its structure, which allows us to investigate it from the perspective of the data flow A with data B being only a nuisance parameter. All conclusions drawn for A should then be equally applicable to B.

A. MAC Phase

Now, we define all formal details. Subscripts A and B denotes variable associated with node A and B respectively. Source data messages are $\mathbf{d}_A$, $\mathbf{d}_B$ and they are composed of data symbols $d_A, d_B \in \mathscr{A}_d = \{0, 1, \ldots, M_d - 1\}$, with alphabet cardinality $|\mathscr{A}_d| = M_d$. For notational simplicity, we omit the sequence number indices of individual symbols. Source node codewords are $\mathbf{c}_A, \mathbf{c}_B$ with code symbols $c_A, c_B \in \mathscr{A}_c$, $|\mathscr{A}_c| = M_c$. The encoding operation is performed by the encoders $\mathscr{C}_A, \mathscr{C}_B$ with codebooks $\mathbf{c}_A \in \mathscr{C}_A$ and $\mathbf{c}_B \in \mathscr{C}_B$. A signal space representation (with an *orthonormal* basis) of the transmitted channel symbols is $s_A = s(c_A)$, $s_B = s(c_B)$, $s_A, s_B \in \mathscr{A}_s \subset \mathbb{C}^N$. We assume a common channel symbol mapper $\mathscr{A}_s(\bullet)$. A signal space representation of the overall coded frame is $\mathbf{s}_A(\mathbf{c}_A)$ and $\mathbf{s}_B(\mathbf{c}_B)$.

We use a slightly relaxed notation for the symbol and the codebook sets and the corresponding mapping and encoding operations. The notation uses the same letter symbol in two different contexts. This simplifies the reader's appreciation of the relatively large number of codebooks/alphabets involved. As an example, the codebook (the set of all codewords) is denoted by $\mathscr{C}$ and it is optionally supplemented by a subscript corresponding to the particular codeword. The notation $\mathscr{C}(\mathbf{d})$ denotes a codeword corresponding to the data input $\mathbf{d}$. A similar notation holds for symbol alphabets, denoted by $\mathscr{A}$, and corresponding mapping operations $\mathscr{A}(\bullet)$.

The received signal is

$$u = s_A + h s_B. \tag{1}$$

It equivalently represents the parametric channel with both links parametrized according to flat fading assumed to be constant over the frame. It is obtained by a proper common rescaling of the true channel response $u' = h_A s_A + h_B s_B$ by $1/h_A$ and denoting $h = h_B/h_A$, $h_A, h_B, h \in \mathbb{C}^1$. The received signal at the relay is

$$x = u + w \tag{2}$$

where the circularly symmetric complex Gaussian noise w has the variance σ_w^2 per complex dimension.

B. BC Phase

The relay receives the signal x and processes it using a Hierarchical Decode and Forward (HDF) strategy. More details will be given in the next section. The output codeword and its code symbols are $\mathbf{c}_R$ and c_R. These are mapped into signal space channel symbols $v \in \mathscr{A}_R$ and signal space codewords $\mathbf{v}$ with the codebook $\mathbf{v} \in \mathscr{C}_R$ and broadcast to destinations A and B. At node B (the destination for data A), the received signal space symbols are

$$y_A = v + w_A \tag{3}$$

where the complex circularly symmetric AWGN w_A has variance σ_A^2 per complex dimension. We denote the signal space symbols at node A (a destination for data B) similarly $y_B = v + w_B$.

III. Layered Hierarchical Exclusive Codebook Design

A. Hierarchical Decode and Forward Strategy

1) Exclusive Law: The HDF strategy is based on relay processing which fully decodes the Hierarchical Data (HD) message $\mathbf{d}_{AB}(\mathbf{d}_A, \mathbf{d}_B)$ and sends out the corresponding codeword $\mathbf{v} = \mathbf{v}(\mathbf{d}_{AB})$ which represents the original data messages $\mathbf{d}_A$ and $\mathbf{d}_B$ only through the *exclusive law* [5]

$$\mathbf{v}(\mathbf{d}_{AB}(\mathbf{d}_A, \mathbf{d}_B)) \neq \mathbf{v}\left(\mathbf{d}_{AB}(\mathbf{d}'_A, \mathbf{d}_B)\right), \ \forall \mathbf{d}_A \neq \mathbf{d}'_A, \tag{4}$$

$$\mathbf{v}(\mathbf{d}_{AB}(\mathbf{d}_A, \mathbf{d}_B)) \neq \mathbf{v}\left(\mathbf{d}_{AB}(\mathbf{d}_A, \mathbf{d}'_B)\right), \ \forall \mathbf{d}_B \neq \mathbf{d}'_B. \tag{5}$$

The hierarchical data are a joint representation of the data from both sources such that they uniquely represent one data source given full knowledge of the other one. Assuming that the destination node B has perfect Side Information (SI) on the node's own data $\mathbf{d}_B$ it can then decode the message $\mathbf{d}_A$ (and similarly for node A). Data $\mathbf{d}_B$ will be called called *complementary* data from the perspective of the data $\mathbf{d}_A$ operations. The SI on the complementary data will be denoted as Complementary SI (C-SI). The exclusive law at the signal space codeword level implies the same for the HD messages

$$\mathbf{d}_{AB}(\mathbf{d}_A, \mathbf{d}_B) \neq \mathbf{d}_{AB}(\mathbf{d}'_A, \mathbf{d}_B), \ \forall \mathbf{d}_A \neq \mathbf{d}'_A, \tag{6}$$

$$\mathbf{d}_{AB}(\mathbf{d}_A, \mathbf{d}_B) \neq \mathbf{d}_{AB}(\mathbf{d}'_A, \mathbf{d}_B), \ \forall \mathbf{d}_B \neq \mathbf{d}'_B. \tag{7}$$

A code (codebook) satisfying the above criteria is called a Hierarchical eXclusive Code (HXC). We will denote the mapping satisfying the exclusive law by the operator $\mathscr{X}(\bullet, \bullet)$.

The hierarchical data can be viewed as the network coded data streams. But notice an important difference. When we simply concatenate independently the NC and then the channel coding, the data generated at the relay are obtained by jointly decoding both individual streams from A and B. The NC mainly serves for the BC stage transmission. The MAC phase is classical joint decoding. In our hierarchical approach this does not happen. The hierarchical data are directly obtained from the received signal observations without the necessity of individual source data decoding. The MAC stage encoding must be such that the observation at the relay allows HXC mapping to the hierarchical data, which may then be encoded into $\mathbf{v}$. In other words, alongside the *complete signal path* (MAC and BC), the coding *must always be HXC* w.r.t. the *hierarchical data*.

2) Relay Hierarchical Codebook Cardinality: In order for the exclusive law to hold, the relay hierarchical codebook cardinality must satisfy $|\mathscr{C}_R| \geq \max(|\mathscr{C}_A|, |\mathscr{C}_B|)$. On the other hand, the cardinality is upper-bounded by all possible $\mathbf{d}_A$ and $\mathbf{d}_B$ message combinations $|\mathscr{C}_R| \leq |\mathscr{C}_A||\mathscr{C}_B|$. The lower bound case requires perfect C-SI on the complementary data at the destination. The upper bound is in fact a classical MAC channel with joint and separate decoding of both data messages at the relay and requires *no* C-SI. Any situation in between those two extreme cases requires partial C-SI at the destination.

The HXC having the minimal cardinality $|\mathscr{C}_R| = \max(|\mathscr{C}_A|, |\mathscr{C}_B|)$ will be called Minimal-HXC (M-HXC). This case puts minimum *throughput* requirements on the BC channel at the price of requiring perfect C-SI at the destination. All codebooks with a higher cardinality will be called Extended-HXC (E-HXC).

B. Layered HXC Design for Perfect C-SI

1) Error Correcting and Hierarchical Exclusive Mapper Layers: It is evident that a direct design of the HXC codebook $\mathscr{C}_R$ providing the mapping $\mathbf{v}(\mathbf{d}_{AB}(\mathbf{d}_A, \mathbf{d}_B))$ is highly complex. An alternative approach is a layered design where the inner layer (closer to the channel symbols) provides the *exclusivity property* whereas the outer layer provides the error correcting (a classical channel coding) functionality. The outer layer code can be an arbitrary state-of-the-art capacity approaching code, e.g. turbo code or LDPC. The dividing line between the layers can have various forms. Both layers can have various levels of the internal structure (memory) and various levels of error correcting capability.

The simplest case is that in which the inner layer is simply a signal space channel symbol memoryless mapper. An alphabet memoryless mapper

$$c_{AB} = \mathscr{X}_c(c_A, c_B) \tag{8}$$

fulfilling the exclusive law will be called a Hierarchical eXclusive Alphabet (HXA). The outer layer then carries all the capacity achieving (error correcting) responsibility. The layered system model is shown in Fig. 2.

2) Symbol-wise Relay HDF Processing: There is also an additional advantage of the layered design. The exclusive property at the symbol level allows a simple determination of the soft *per-symbol* measure decoding metric for hierarchical symbols at the relay. It can either be directly used by the full relay decoder or properly source encoded and sent on a per-symbol basis without decoding. The latter has the major advantage of lowering the latency of the relay processing. The topic of per-symbol *only* relay processing is out of this paper's scope.

3) Layered Design: The layered design relies on the theorems described in detail in paper [3] and also in [10]. Here, we only state the results. The layered HXC design is possible and the capacity is achieved by an outer standard channel code $\mathscr{C}_A = \mathscr{C}_B = \mathscr{C}$ combined with inner symbol-wise HXA $\mathscr{A}_s$. We will refer to this as the *layered HXC design theorem*. The theorem assumes the minimal HXA mapping for the data and generally the multi-symbol HXA mapping for constellation space symbols received by the relay $c_{AB} = \mathscr{X}_c(c_A, c_B)$ where $u(c_A, c_B) = \mathscr{A}_u(c_{AB})$. The capacity is alphabet constrained by *H-MAC channel hierarchical symbols* and the H-MAC rate region has a rectangular shape

$$R_A = R_B = R_{AB} \leq I(c_{AB}; x). \tag{9}$$

4) Equal Information Rates and Uniform-Input Capacity: For the present, we assume equal data information rates $R_A = R_B$ with common shared codebook and mappers. A solution of the asymmetric case is not dealt with here.

The mutual information $I(c_{AB}; x)$ is not a capacity in a strict sense since we do not maximize over the input distribution. We will however assume that this distribution is given and it is a uniform distribution of a discrete channel input. This will be called a *uniform-input (alphabet constrained) capacity*. We denote it by the letter C however the uniform-input assumption must be kept in mind.

C. Relay Hierarchical Decoding with Soft HXA Metric and Common Codebook

The soft-output H-MAC relay demodulator produces the symbol-wise metric $\mu(c_{AB})$ on the hierarchical data c_{AB} which is fed into the hierarchical H-MAC stage decoder $\mathscr{D}_{AB}$. The metric must properly reflect the fact that the useful signal $\mathscr{A}_u(c_{AB})$ is a set (class) with the cardinality generally higher than one and also depending on the channel parametrization. No particular form of metric evaluation, nor its utilization by the decoder, is dealt with here.

IV. Throughput Rate Region

A. Hierarchical MAC Rate Region

1) Hierarchical Mutual Information: It follows from the Layered HXC Design theorem that the throughput rate region is rectangular and given by the hierarchical mutual information $I(c_{AB}; x)$. It is evaluated for given chosen symbol alphabets $\mathscr{A}_s$ with given channel parametrization

$$x = u(s(c_A) + hs(c_B)) + w. \tag{10}$$

The hierarchical code symbols are mapped on the useful signal by $u(c_A, c_B) = \mathscr{A}_u(c_{AB})$. The transmitted signals, the useful H-MAC signal and the received signal are signal space

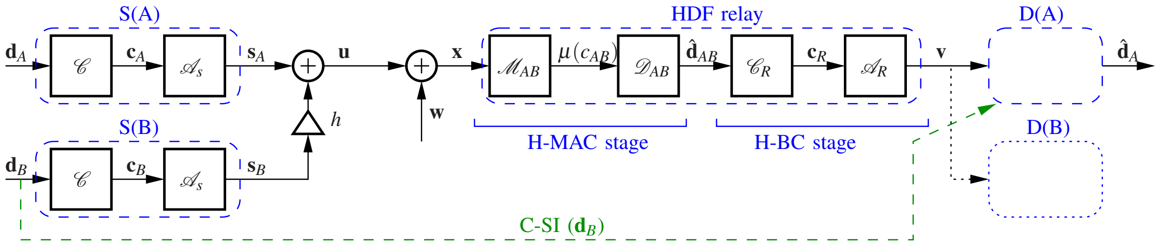

Figure 2. The system model for layered HXC design.

representations $s(c_A), s(c_B), u, x \in \mathbb{C}^N$. The mutual information evaluation must respect the fact that the $\mathscr{A}_u(c_{AB})$ is generally a set (class) of multiple possible symbols for each particular c_{AB}.

The hierarchical mutual information (uniform-input capacity) is

$$C_{AB} = I(c_{AB};x) = \mathscr{H}[x] - \mathscr{H}[x|c_{AB}] \quad (11)$$
$$= \mathscr{H}[x] - \mathscr{H}[x|\mathscr{A}_u(c_{AB})]. \quad (12)$$

The channel parametrization h and symbol space alphabets $s(\bullet)$ are *implicitly* hidden in the hierarchical class $\mathscr{A}_u(c_{AB})$ definition and we do not use an explicit notation for that.

2) Received Signal Entropy: From the entropy definition, we get (all integrals are over $\mathbb{C}^N$ support)

$$\mathscr{H}[x] = -\int p(x) \lg p(x) \, \mathrm{d}x \quad (13)$$

where the PDF is

$$p(x) = \frac{1}{M_c^2} \sum_{c_A, c_B} p_w(x - u(c_A, c_B)) \quad (14)$$

and $p_w(w)$ is the PDF of complex signal space AWGN representation

$$p_w(w) = \frac{1}{(\pi \sigma_W^2)^N} \exp\left(-\frac{\|w\|^2}{\sigma_w^2}\right). \quad (15)$$

We assumed uniformly distributed code symbols $c_{A,}c_B$.

3) Conditional Entropy: An evaluation of the conditional entropy $\mathscr{H}[x|c_{AB}] = \mathscr{H}[x|\mathscr{A}_u(c_{AB})]$ is slightly more complicated compared to the classical point-to-point single user channel. Generally, it is *not* the entropy of the AWGN $\mathscr{H}[x|c_{AB}] \neq \mathscr{H}[w]$ as it would have been in the single user channel. This is because, given c_{AB}, the actual useful signal is a multi-symbol class $\mathscr{A}_u(c_{AB})$.

From the definition, we get

$$\mathscr{H}[x|c_{AB}] = -\sum_{c_{AB}} \Pr\{c_{AB}\} \int p(x|c_{AB}) \lg p(x|c_{AB}) \, \mathrm{d}x. \quad (16)$$

In the case of a *minimal* exclusive code (M-HXC) and uniformly distributed c_A, c_B, the hierarchical symbols have $\Pr\{c_{AB}\} = 1/M_c$. The conditional PDF is

$$p(x|c_{AB}) = \frac{1}{M_c} \sum_{c_A, c_B : \mathscr{X}_c(c_A, c_B) = c_{AB}} p_w(x - u(c_A, c_B)). \quad (17)$$

We sum only over such pairs c_A, c_B that are compliant with the given hierarchical symbol c_{AB}. There are M_c such uniformly probable cases for a *minimal* exclusive code and equal cardinality symbols. A form of the result in alternative notation can be obtained using the indicator (Kronecker delta) function $\mathscr{I}_c(c_A, c_B, c_{AB}) = \delta[c_{AB} - \mathscr{X}_c(c_A, c_B)]$

$$p(x|c_{AB}) = \frac{1}{M_c} \sum_{c_A, c_B} p_w(x - u(c_A, c_B)) \mathscr{I}_c(c_A, c_B, c_{AB}). \quad (18)$$

B. MAC and Cut-Set Bound Reference

As a reference case, we consider the classical MAC rate region 1st and 2nd order limits $I(c_A;x|c_B), I(c_A, c_B;x)$. Notice that the first order rate limit $I(c_A;x|c_B)$ is in fact a cut-set bound [11] with finite alphabet limitation. The second order limit $I(c_A, c_B;x)$ corresponds to the joint decode and forward strategy where individual data streams from A and B are separately decoded at the relay.

For comparison, we evaluate $I(c_A;x|c_B)$ as a limiting performance criterion. The value R_0 is in fact the mutual information of the single user A channel as if there is no user B at all. Notice that this can be quite easily modelled by setting the channel B transfer $h = 0$. The uniform-input alphabet constrained capacity is found as

$$C_0 = I(c_A;x|c_B) = \mathscr{H}[x|c_B] - \mathscr{H}[x|c_A, c_B] \quad (19)$$

where the second term is simply the entropy of AWGN

$$\mathscr{H}[x|c_A, c_B] = \mathscr{H}[w] = N \lg\left(\pi e \sigma_w^2\right). \quad (20)$$

C. Unconstrained Capacity

We also include the true alphabet unconstrained cut-set bound capacity for given input variance which is

$$C_u = \lg\left(1 + \mathrm{E}\left[\|\mathbf{s}_A\|^2\right] / \sigma_w^2\right). \quad (21)$$

D. Monte-Carlo Evaluation of the Integrals

All integrals involved in the entropy evaluations are complicated multi-fold integrals over the N-dimensional complex plane. Their direct evaluation by traditional numerical integration procedures is known to be complex and very slowly converging. Better computational efficiency is achieved by Monte-Carlo numerical evaluation of the integrals. This is not to be confused with Monte-Carlo simulation. We use it for the numerical evaluation of the integrals, not for running the system with some particular coding and signal processing algorithms. The method is usable for evaluation of any summation or integration of the form $I = \int f(x) g(x)\, dx$. We consider $f(x)$ as a PDF, and hence the integral can be evaluated as $I = \mathrm{E}_{f(x)}[g(x)]$. Providing that we are able to generate random values with the distribution $f(x)$: $x \sim f(x)$ then the integral is approximated by evaluating the empirical mean $\hat{\mathrm{E}}[\centerdot]$ of $g(x)$. Generation of the random variable with given a density is a relatively easy task particularly for linear Gaussian mixture densities. We can also always resort to building a full system model producing the desired random variable and feeding that system with its all random excitations — typically data and noise sources.

A numerical evaluation of $\mathscr{H}[x]$ gives

$$\mathscr{H}[x] \approx -\hat{\mathrm{E}}_{x \sim p(x)} \left[\lg \left(\frac{1}{M_c^2} \sum_{c_A, c_B} p_w(x - u(c_A, c_B)) \right) \right] \qquad (22)$$

where x is generated with the distribution (14). The conditional entropy approximation is obtained as

$$\mathscr{H}[x|c_{AB}] = -\hat{\mathrm{E}}_{x, c_{AB} \sim p(x, c_{AB})} \left[\lg p(x|c_{AB}) \right]. \qquad (23)$$

V. Numerical Results

The hierarchical rate and single-user (alphabet limited cut-set bound) rates were evaluated for a number of signal alphabets $\mathscr{A}_s$ and channel parametrizations. The Signal-to-Noise Ratio (SNR) is defined as the ratio of the real base-band symbol energy of one source (e.g. A, to have a fair comparison for reference cases) to the noise power spectrum density ratio $\gamma = \left(\bar{\mathscr{E}}_{\mathbf{s}_A}/2 \right) / N_0$. Assuming orthonormal basis signal space complex envelope representation of the AWGN, we have $\sigma_w^2 = 2N_0$ and thus $\gamma = \mathrm{E}\left[\|\mathbf{s}_A\|^2 \right] / \sigma_w^2$. The alphabet $\mathscr{A}_s$ is indexed by symbols $c_A, c_B \in \{0, \ldots, M_c - 1\}$. The exclusive hierarchical mapping is $c_{AB} = c_A \oplus c_B$.

Figs. 3, 4 and 5 show comparisons of the hierarchical symmetric capacity with various channel parametrizations with the alphabet constrained cut-set bound and the unconstrained AWGN cut-set bound capacity. Fig. 6 shows the hierarchical capacity outage probability assuming uniform random channel phase. The amplitude is kept constant ($|h| = 1$) in our setup to respect the symmetry of the rates from A and B. This performance measure is suitable for the system behaviour with a packet based transmission and a quasi-static channel. An evaluation of the outage capacity allows us to quantitatively measure the capacity variance (dispersion) that results from its dependance on the channel parametrization. The lower the variance (the higher slope of the curve) the better, i.e. the parametrization has lower effect on the capacity.

VI. Discussion and Conclusions

The numerical evaluation shows several important performance phenomena of the HDF strategy.

First, we show that the performance of the layered design can closely approach the alphabet constrained cut-set bound limit for medium to high SNR. The main advantage of the layered design is its capability for reuse the standard single-link capacity approaching codes (turbo code, LDPC).

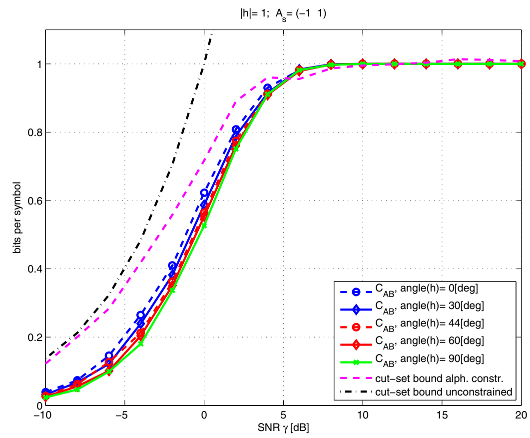

Figure 3. A comparison of capacities C_{AB}, C_0, C_u for BPSK alphabet and various channel parametrizations.

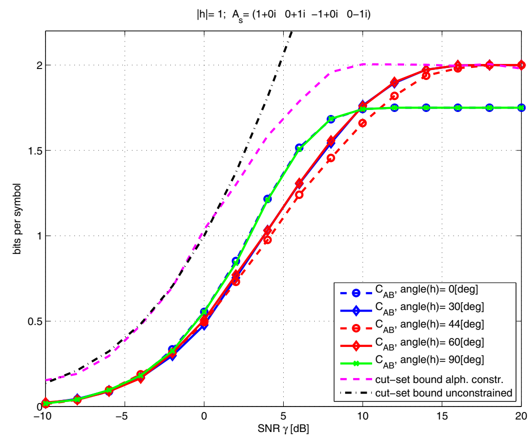

Figure 4. A comparison of capacities C_{AB}, C_0, C_u for QPSK alphabet with indexing #1 and various channel parametrizations.

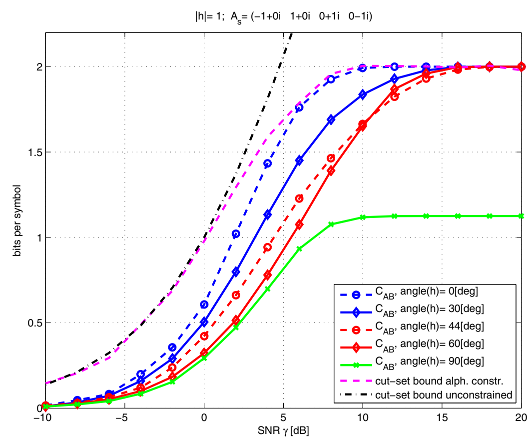

Figure 5. A comparison of capacities C_{AB}, C_0, C_u for QPSK alphabet with indexing #2 and various channel parametrizations.

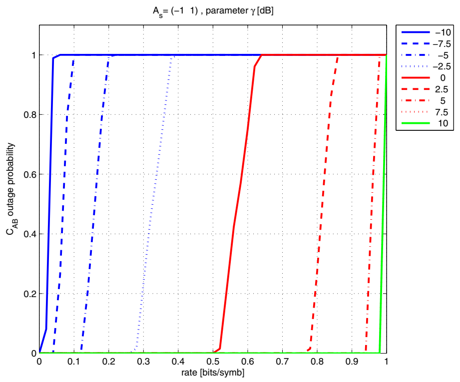

Figure 6. Hierarchical capacity outage probability $\Pr\{C_{AB} < R_{AB}\}$ for BPSK component constellation.

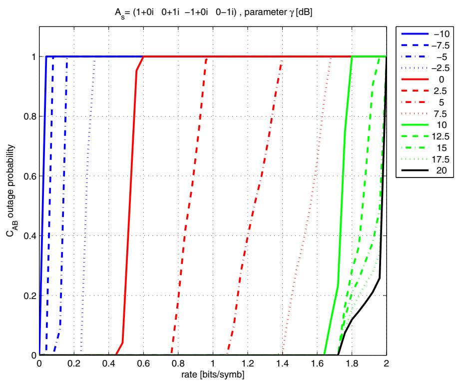

Figure 7. Hierarchical capacity outage probability $\Pr\{C_{AB} < R_{AB}\}$ for QPSK (indexing #1) component constellation.

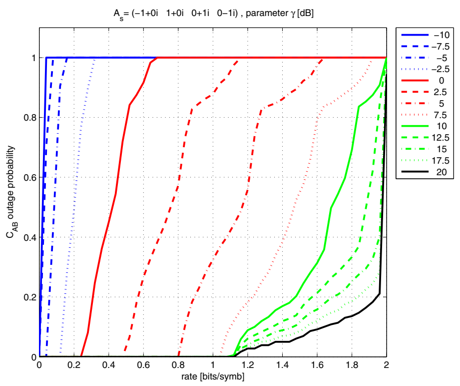

Figure 8. Hierarchical capacity outage probability $\Pr\{C_{AB} < R_{AB}\}$ for QPSK (indexing #2) component constellation.

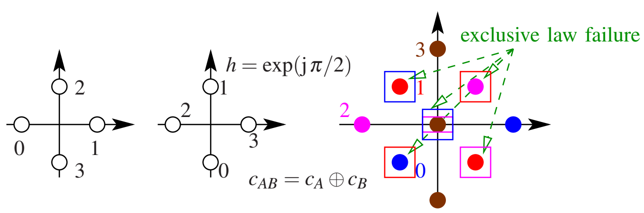

Figure 9. An example of the exclusive law failure for QPSK (indexing #2) and specific phase rotation.

The second phenomenon is the inherent impact of the channel parametrization on the achievable rates. Different constellations and even the same constellation with different indexing are vulnerable to different extents. The reason for this is two-fold: (a) the distances among the points of the hierarchical constellation $\mathscr{A}_u(c_{AB})$ are distorted by the parametrization, and (b), some specific parameter values have disastrous effects. Some channel rotations cause the points corresponding to different hierarchical codewords to fall into the same useful signal u and in fact, the useful signal is *no longer exclusive*.

BPSK appears to be relatively resistant to these effects of the parametrization. We can only observe the effect of the $\mathscr{A}_u(c_{AB})$ distance distortions for different phase shifts. QPSK however, in both indexing variants, apart from small distance distortions, exhibits strongly the effect of the critical phase rotation causing failure of the exclusivity law (Fig. 9). We also see that different QPSK indexing changes the vulnerability of performance to these effects. QPSK with indexing #2 demonstrates both larger and more frequent performance degradations. This can be seen in the lower slope of the capacity outage curves $\Pr\{C_{AB} < R_{AB}\}$. The lower slope shows that there is a wider range of different capacities for a given fixed SNR when considering all random uniform phase rotations.

The results of the paper highlight the importance of code design that properly takes the channel parametrization into an account, e.g. by being invariant to this. Signal phase predistortion to avoid such situations is feasible only in very specific situations, e.g. for channels which are invariant for several frames, allowing channel state information to be transmitted to both sources. It is not generally suitable for short packet oriented transmissions.

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