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Abstract—**A CPM type constant envelope space-time modulation is an attractive option especially for very high frequency and mobile applications due to its resistance to a nonlinear distortion. The paper analyzes the mean squared distance of the trellis coded CPM type constant envelope space-time modulated signal in a Rayleigh slowly flat fading spatial diversity channel with independent coefficients. It is shown that the distance evaluation depends on both—the modulator trellis and on the distance evaluation trellis. The latter has a special properties regarding to its free paths. These properties—a critical path and the distance increments behavior are the basis for the space-time trellis code design. We identify the trellis code (data to channel phase symbols mapping) design rules minimizing the mean squared distance. The critical path is shown to be the determining factor.**

I. Introduction

Spatial diversity communication is currently one of the most targeted research topics. The most attractive is its potential for increasing the channel capacity in the bandwidth restricted environment. The topic is addressed from number of different viewpoints starting with the channel capacity evaluation (see e.g. [1], [2]) and going into the synthesis of the space-time modulation (e.g. [3]). However almost all resources have focused so far only on *linear* modulation schemes. Reference [4] belongs among rare exceptions. The *nonlinear constant envelope* modulations provide an attractive option from the implementation point of view especially on very high frequencies for indoor communication or for mobile devices with restricted power resources. Especially the resistance of such modulations to nonlinear distortion is very appealing. This paper is a follow up of the on going work started in [5]. The paper contains extended version of the derivations from [5] supplemented by additional results and correcting unfortunate numerical mistakes in the example application code of the [5].

We will focus on the problem by analyzing the distance properties of constant envelope space-time modulation and then by formulating the design rules for a trellis code maximizing the distance in Rayleigh spatial diversity channel with independent paths. Here, we will formulate the rules in terms of *channel phase symbols* properties unlike as it is developed in [4] where the rules are related to the transmitted signal itself. Our approach is more suitable for the design of the trellis code, i.e. data to channel phase symbols mapping.

The paper considers a general nonlinear constant envelope modulator with a finite correlation length. The nonlinearity of the modulation and the inherent memory of its phase expansion part inevitably makes the treatment more complicated then in the case of linear modulation. It is known that a class of binary constant envelope phase modulations can be equivalently expressed as a linear-like modulation using Laurent expansion (see [6]). However we will not proceed in this way in the paper. First because of the binary data constrain and also because of a lack (except for unity correlation length) of straightforward relation of the channel phase symbols to the data. This disqualifies the approach for the trellis code synthesis.

II. System Model

A space-time (ST) modulated signal with constant envelope components is a vector signal[1] $\mathbf{s}(t) = [s_1(t), \ldots, s_{N_t}(t)]^T$ where N_t is the number of transmitter antennas. Each component is CPM-class modulated signal

$$s_i(t) = e^{j2\pi\kappa \sum_n q_{n,i}\beta(t-nT_S)} \quad (1)$$

where κ is a positive real valued constant—modulation index, $d_n \in \{0, 1, \ldots (M_d - 1)\}$ is the data symbol, $\sigma_n \in \{0, 1, \ldots (M_\sigma - 1)\}$ is the state of the modulator and T_S is the symbol period. The whole data message is $\mathbf{d} = [\ldots, d_n, \ldots]^T$. Variables $q_{n,i} = q_{n,i}(d_n, \sigma_n)$ are discrete *channel phase symbols*. Function $\beta(t)$ is a phase function (common to all i) obeying $\beta(t) = 0,\ t \le 0$ and $\beta(t) = 1/2,\ t \ge LT_S$. We also define $\mathbf{q}_n = [q_{n,1}, \ldots q_{n,N_t}]^T$ and $\Phi_{n,i} = q_{n,i}\beta(t - nT_S)$ Modulator state σ_n obeys a state transition equation $\sigma_{n+1} = \sigma(d_n, \sigma_n)$. We assume the

[1] All vectors are column vectors. All derivations are carried out for complex envelope signals.

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same modulation index κ in all components of the signal. We also assume the unity and constant amplitude of the modulated signal over all components. In the next, we restrict our attention to the phase modulation functions with a finite correlation length $LT_S,\ L \in \mathbb{N}$.

The mapping between data and channel phase symbols $\{\mathbf{d}\} \mapsto \{\mathbf{q}\}$ represents the *trellis code* for the *constant envelope ST modulation*. The trellis code is a mapping between data and *channel phase signal space*. This needs to be distinguished from the case of linear ST modulation where the channel symbols are direct expansion of the modulated signal in the signal space and therefore the trellis code is a mapping between data and signal space expansion of the modulated signal.

The signal passes through the AWGN spatial diversity channel. A received signal on the k-th receive antenna is

$$x_k(t) = u_k(t) + w_k(t), \quad k \in \{1, \ldots, N_r\} \qquad (2)$$

where N_r is the number of the receiver antennas. Noise components w_k are assumed to be complex white Gaussian IID[2] processes each with IID real and imaginary parts. A double-sided power spectrum density of w_k is $2N_0$ for all k. A useful part of the received signal on the k-th receive antenna is

$$u_k(t) = \sum_{i=1}^{N_t} a_{ik} s_i(t), \quad k \in \{1, \ldots, N_r\}. \qquad (3)$$

The channel is considered to be a slowly flat fading one. The coefficients a_{ik} denote the complex valued transfer from the i-th transmitter to the k-th receiver. They are zero mean complex Gaussian IID random variables assumed to be constant during the data message transfer.

III. Mean squared distance evaluation trellis

A. Mean squared distance

In this section, we derive an expression for the Euclidean distance for useful received signals corresponding to two different data messages $\mathbf{d}^{(1)}$ and $\mathbf{d}^{(2)}$. We denote the corresponding modulated signal components

$$s_i^{(1)} = e^{j2\pi\kappa \sum_n \Phi_{n,i}^{(1)}}, \quad s_i^{(2)} = e^{j2\pi\kappa \sum_n \Phi_{n,i}^{(2)}} \qquad (4)$$

and similarly $u_k^{(1)}(t), u_k^{(2)}(t)$. For the distance evaluation, the received signals $u_k(t)$ can be conveniently and *equivalently* organized in a sequence instead as a vector in parallel. This allows a simple expression for the squared distance of the signals conditioned by particular values a_{ik} observed at the receiver

$$\rho_a^2 = \sum_{k=1}^{M_r} \int_{-\infty}^{\infty} \left| u_k^{(1)} - u_k^{(2)} \right|^2 dt = \sum_{k=1}^{N_r} \sum_{i=1}^{N_t} \sum_{i'=1}^{N_t} a_{ik} a_{i'k}^* V_{ii'} \qquad (5)$$

where

$$V_{ii'} = \int_{-\infty}^{\infty} (s_i^{(1)} - s_i^{(2)})(s_{i'}^{(1)} - s_{i'}^{(2)})^* \, dt. \qquad (6)$$

Defining $\mathbf{a}_k = [a_{1k}, \ldots, a_{M_t k}]^T$, $[\mathbf{A}]_{\mathbf{ik}} = \mathbf{a_{ik}}$ and $\mathbf{V}$ as the matrix with elements $V_{ii'}$ in i-th row and i'-th column, we can express this in a matrix form by[3]

$$\rho_a^2 = \sum_{k=1}^{N_r} \mathbf{a}_k^T \mathbf{V} \mathbf{a}_k^* = \sum_{k=1}^{N_r} \mathrm{tr}(\mathbf{a}_k \mathbf{a}_k^H \mathbf{V}^T) = \mathrm{tr}\left(\mathbf{A}^T \mathbf{V} \mathbf{A}^*\right). \qquad (7)$$

Probability of the pairwise message error between transmitted message $\mathbf{d} = \mathbf{d}^{(1)}$ and the message $\mathbf{d}^{(2)}$ examined by the detector is for AWGN channel and for given particular a_{ik}

$$\begin{aligned} P_{2e,a} &= \Pr\{\rho_{xa}(\mathbf{x}, \mathbf{d}^{(2)}) < \rho_{xa}(\mathbf{x}, \mathbf{d}^{(1)}) | \mathbf{d} = \mathbf{d}^{(1)}\} \\ &= Q\left(\frac{\rho_a}{2\sqrt{N_0}}\right) \end{aligned} \qquad (8)$$

where $\rho_{xa}(\mathbf{x}, \mathbf{d})$ is the Euclidean detector metric for the data $\mathbf{d}$ and the received signal $\mathbf{x}$ assuming a perfect channel state information knowledge. $Q(.)$ is the normalized right-tail Gaussian probability function. This probability can be (except for small argument values) upper bounded by

$$P_{2e,a} < P_{2e,a}^{(\mathrm{ub})} = \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{\rho_a^2}{8N_0}\right). \qquad (9)$$

In order to get an upper bound on the average probability of the pairwise error we would have to perform the averaging over all values of a_{ik}, $P_{2e}^{(\mathrm{ub})} = \mathrm{E}[P_{2e,a}^{(\mathrm{ub})}]$. However, unlike for the case of a linear modulation treated for example in [3], this expression is mathematically difficult to further manipulate for our case of the *nonlinear* modulation when a simple relation to $q_{n,i}$ is required for the code synthesis. Instead, we average directly the squared distance $\rho^2 = \mathrm{E}[\rho_a^2]$. This value can be then used in the exponential to obtain an approximation of the upper bound

$$P_{2e}^{\prime(\mathrm{ub})} = \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{\rho^2}{8N_0}\right). \qquad (10)$$

Of course there is no simple analytical relation between $P_{2e}^{(\mathrm{ub})}$ and $P_{2e}^{\prime(\mathrm{ub})}$. However, we can *physically interpret* what is achieved by *maximizing* averaged ρ^2. This is in fact equivalent to minimizing $\mathrm{E}[\ln P_{2e,a}^{(\mathrm{ub})}]$ which can be easily seen from (9). A choice of the pairwise error probability logarithm minimization seems to be quite acceptable especially realizing that the mutual information and the channel capacity are closely related to the expectation of the error probability logarithm. Later in the paper, we

[2] Independent Identically Distributed.

[3] It holds $\mathbf{x}^T \mathbf{y} = \mathrm{tr}(\mathbf{x}\mathbf{y}^T)$ for any equal length vectors. The function tr(.) denotes a trace of the matrix.

will support this step by simulation results relating mutually $P_{2e}^{\prime(\mathrm{ub})}$ and $P_{2e}^{(\mathrm{ub})}$ values.

Now we evaluate the average useful received signal squared distance over all values of a_{ik}. The averaging is performed over large number of data transfer sessions. Considering the assumption of slow fading together with some kind of splitting the long message into smaller pieces, this also corresponds to the averaging the distance over the long data message split into independent smaller pieces. The average squared distance is

$$\rho^2 = \mathrm{E}[\rho_a^2] = \sum_{k=1}^{N_r} \mathrm{tr}(\mathbf{R}_{\mathbf{a}_k} \mathbf{V}^T) \tag{11}$$

where $\mathbf{R}_{\mathbf{a}_k} = \mathrm{E}[\mathbf{a}_k \mathbf{a}_k^H]$. Using the assumption of IID random variables a_{ik}, we get $\mathbf{R}_{\mathbf{a}_k} = \sigma_a^2 \mathbf{I}$ and subsequently

$$\rho^2 = \sum_{k=1}^{N_r} \mathrm{tr}(\sigma_a^2 \mathbf{I} \mathbf{V}^T) = \sigma_a^2 \sum_{k=1}^{N_r} \mathrm{tr}(\mathbf{V}^T) = \sigma_a^2 N_r \, \mathrm{tr}(\mathbf{V}). \tag{12}$$

The trace of the matrix $\mathbf{V}$ is $\mathrm{tr}(\mathbf{V}) = \sum_{i=1}^{N_t} V_{ii}$ where

$$V_{ii} = \int_{-\infty}^{\infty} |s_i^{(1)}|^2 + |s_i^{(2)}|^2 - 2\Re[s_i^{(1)} s_i^{(2)*}] \, dt. \tag{13}$$

A substitution of the modulated signal (4) into this expression and a normalization of the squared distance (12) by a suitable constant $\rho'^2 = \rho^2/(2\sigma_a^2 N_r T_S)$ gives

$$\begin{aligned} \rho'^2 &= \sum_{i=1}^{N_t} \frac{1}{T_S} \int_{-\infty}^{\infty} 1 - \Re\left[e^{j2\pi\kappa \sum_n (\Phi_{n,i}^{(1)} - \Phi_{n,i}^{(2)})} \right] dt \\ &= \sum_{i=1}^{N_t} \sum_{m=-\infty}^{\infty} \Delta\rho_i'^2(m) \end{aligned} \tag{14}$$

where the squared distance increments for the i-th branch are

$$\Delta\rho_i'^2(m) = 1 - \frac{1}{T_S} \int_{mT_S}^{(m+1)T_S} \Re\left[e^{j2\pi\kappa \sum_n (\Phi_{n,i}^{(1)} - \Phi_{n,i}^{(2)})} \right] dt. \tag{15}$$

Notice that ρ'^2 is the normalized value *per one receiver antenna*. A substitution for phase modulation functions and utilization of the finite correlation length assumption gives

$$\Delta\rho_i'^2(m) = 1 - \Re[\eta_i(m)\lambda_i(m)] \tag{16}$$

where

$$\eta_i(m) = \exp\left(j2\pi\kappa \sum_{n=-\infty}^{m-L} \frac{\Delta q_{n,i}}{2} \right) \tag{17}$$

and

$$\lambda_i(m) = \frac{1}{T_S} \int_0^{T_S} e^{j2\pi\kappa \sum_{n=-L+1}^{0} \Delta q_{m+n,i} \beta(t-nT_S)} \, dt. \tag{18}$$

The quantity $\Delta q_{m,i} = q_{m,i}^{(1)} - q_{m,i}^{(2)}$ is the channel phase symbol difference. An overall squared distance increment reflecting all transmitted signals is

$$\Delta\rho'^2(m) = \sum_{i=1}^{N_t} \Delta\rho_i'^2(m) = N_t - \sum_{i=1}^{N_t} \Re\left[\eta_i(m)\lambda_i(m)\right]. \tag{19}$$

If we consider a *special case* of high practical importance when the modulation index is a rational number $\kappa = \kappa_1/\kappa_2$, $\kappa_1, \kappa_2 \in \mathbb{N}$ and when channel phase symbols are drawn from the set $q_{m,i} \in \{\pm1, \pm3, \ldots, \pm(M_q - 1)\}$ then the differences are $\Delta q_{m,i} \in \{0, \pm2, \pm4, \ldots, \pm2(M_q - 1)\}$ and $\sum_{n=-\infty}^{m-L} \Delta q_{n,i}/2$ becomes an integer. The set of all possible $\eta_i(m)$ becomes finite $\eta_i(m) \in \{\eta^{(p)}\}_{p=1}^{M_\eta}$. We can define

$$\psi_i(m) = \left(M_\eta \kappa \sum_{n=-\infty}^{m-L} \frac{\Delta q_{n,i}}{2} \right) \bmod M_\eta \tag{20}$$

($\psi_i(m) \in \{0, 1, \ldots (M_\eta - 1)\}$) which is related to $\eta_i(m)$ by

$$\eta_i(m) = \exp\left(j2\pi \frac{\psi_i(m)}{M_\eta} \right). \tag{21}$$

We will assume this in the whole subsequent treatment.

B. Distance evaluation trellis

We can now define $\mathbf{\Theta}(m) = [\boldsymbol{\theta}_1(m), \ldots, \boldsymbol{\theta}_{N_t}(m)]$

$$\boldsymbol{\theta}_i(m) = [\psi_i(m), \Delta q_{m-L+1,i}/2, \ldots, \Delta q_{m-1,i}/2]^T. \tag{22}$$

It can be easily seen that $\mathbf{\Theta}(m)$ and $\Delta\mathbf{q}_m = \mathbf{q}_m^{(1)} - \mathbf{q}_m^{(2)}$ can be interpreted as a *state* and an *input* of the distance evaluation procedure which itself possesses the memory. Compare this result with a typical trellis code design for linear modulations where the only memory needed to be considered is the one of the modulator trellis. The distance evaluation in that case is memoryless. The memory of the distance evaluation process is just the one that makes the modulator trellis code synthesis difficult for the nonlinear modulation.

The squared distance increment $\Delta\rho'^2(m)$ can be calculated having knowledge of the input $\Delta\mathbf{q}_m$ and the state $\mathbf{\Theta}(m)$. The quantity $\mathbf{\Theta}(m)$ will be referred as a *distance evaluation state*. A state transition equation $\boldsymbol{\theta}_i(m+1) = \boldsymbol{\theta}(\boldsymbol{\theta}_i(m), \Delta q_{m,i})$ for i-th branch is clearly[4]

$$\begin{aligned} [\boldsymbol{\theta}_i(m+1)]_1 &= ([\boldsymbol{\theta}_i(m)]_1 + M_\eta \kappa [\boldsymbol{\theta}_i(m)]_2) \bmod M_\eta \\ [\boldsymbol{\theta}_i(m+1)]_2 &= [\boldsymbol{\theta}_i(m)]_3 \\ &\vdots \\ [\boldsymbol{\theta}_i(m+1)]_{L-1} &= [\boldsymbol{\theta}_i(m)]_L \\ [\boldsymbol{\theta}_i(m+1)]_L &= \Delta q_{m,i}/2. \end{aligned} \tag{23}$$

[4] $[\mathbf{x}]_i$ denotes i-th component of the vector.

The number of all possible distance evaluation states is

$$M_{\mathbf{\Theta}} = ((2M_q - 1)^{L-1} M_\eta)^{N_t}. \quad (24)$$

The distance evaluation can be conveniently graphically represented by a path in a *distance evaluation trellis*. It is important to stress that the distance evaluation states and trellis are completely independent with the states and trellis of the modulator. The distance evaluation trellis serves as the distance evaluation tool. The particular form of the distance evaluation trellis depends only on the channel phase symbols difference and on κ, N_t and $\beta(t)$. It is not directly related to the actual message data (only through $\Delta\mathbf{q}_m$).

The distance evaluation trellis will be used later for an evaluation of the free distance and for a code design maximizing this free distance. From that point of view, the initial distance evaluation state is always $\boldsymbol{\theta}_i(0) = [0, 0, \ldots, 0]^T$, $i = 1, \ldots, N_t$ at the point[5] where the path in the modulator trellis splits for the first time for two different data messages $\mathbf{d}^{(1)}, \mathbf{d}^{(2)}$. This directly follows from the fact that the data and consequently the channel phase symbols are the same up to the point of the split ($m = 0$) for both messages and thus $\Delta q_{m,i} = 0$ for $m < 0$.

C. Example case

Now we present an example case as an illustration of the distance evaluation trellis concept. The phase function is

$$\beta(t) = \int_{-\infty}^{t} \mu(\tau)\, d\tau \quad (25)$$

where $\mu(\tau)$ is the RC2 frequency impulse[6] (i.e. $L = 2$) and $q_{n,i} \in \{\pm 1\}$, $M_q = 2$ and $\kappa = 1/2$. We will consider the case of 2-space code (i.e. two transmitter antennas $N_t = 2$). These particular values imply also that $\Delta q_{n,i} \in \{0, \pm 2\}$ and $\psi_i(m) \in \{0, 1\}$, $M_\eta = 2$. The distance evaluation state vector is

$$\boldsymbol{\theta}_i(m) = [\psi_i(m), \Delta q_{m-1,i}/2]^T \quad (26)$$

with initial state $\boldsymbol{\theta}_i(0) = [0, 0]^T$. The total number of distance evaluation states is $M_{\mathbf{\Theta}} = 6^2 = 36$. In this example case, the first equation in (23) becomes particularly simple

$$[\boldsymbol{\theta}_i(m+1)]_1 = ([\boldsymbol{\theta}_i(m)]_1 + [\boldsymbol{\theta}_i(m)]_2) \bmod 2. \quad (27)$$

There is one special feature of the case with $M_\eta = 2$ which is however *not* generally applicable for other values. This is the symmetry of the previous equation with respect to the sign of the $[\boldsymbol{\theta}_i(m)]_2$. We can therefore define a new state with reduced number of possible values

$$\boldsymbol{\theta}'_i(m) = [\psi_i(m), |\Delta q_{m-1,i}|/2]^T, \quad (28)$$

$$\boldsymbol{\theta}'_i(m) \in \{[0,0]^T, [0,1]^T, [1,0]^T, [1,1]^T\}. \quad (29)$$

The number of all possible states is then reduced to $M_{\mathbf{\Theta}'} = 4^2 = 16$.

For this particular example, we can evaluate mutual relation between $P_{2e}^{\prime(\mathrm{ub})}$ and $P_{2e}^{(\mathrm{ub})}$ values by means of simulation. We plot a 2-dimensional graph $P_{2e}^{\prime(\mathrm{ub})}$ versus $P_{2e}^{(\mathrm{ub})}$ where each point corresponds to one particular pairwise error probability event corresponding to particular pair of channel phase symbols sequences $\vec{\mathbf{q}}^{(1)}$ and $\vec{\mathbf{q}}^{(2)}$. We plot the point for all possible combinations of $\vec{\mathbf{q}}^{(1)}$ and $\vec{\mathbf{q}}^{(2)}$. The results for various parameters are on Figure 1. Signal to noise ratio is defined as

$$\gamma_S = \frac{\mathrm{E}\left[\sum_{k=1}^{N_r} |u_k|^2\right]}{N_0} = \frac{N_t N_r \sigma_a^2}{N_0}. \quad (30)$$

The most important is the behavior of relation in the *north-east* part of the graph. This is the region which dominates the overall error properties of the detection. We can see that the relation of $P_{2e}^{\prime(\mathrm{ub})}$ and $P_{2e}^{(\mathrm{ub})}$ is close to a *monotone* function in that region. This justifies usage of $P_{2e}^{\prime(\mathrm{ub})}$ as a code design objective function.

IV. Modulator Trellis Code Design

A. Design objectives

In this section, we attempt to synthesize a modulator coder—i.e. to design a mapping[7] of data symbols d_n and modulator states σ_n on the channel phase symbols $\mathbf{q}_n$ and modulator state transitions $\sigma_{n+1} = \sigma(d_n, \sigma_n)$ in such a way that the minimum mean squared distance over all different data messages

$$\rho'^2_{\mathrm{free}} = \min_{\mathbf{d}^{(1)} \neq \mathbf{d}^{(2)}} \rho'^2 \quad (31)$$

is maximized. The squared distance ρ'^2 evaluation is however (as derived in the previous section) also ruled by the state-space model (state evaluation trellis). This makes the synthesis particularly difficult since we need to design a system with memory (discrete part of the modulator) in such a way that its output difference $\Delta\mathbf{q}_n$ for two data messages fed into another system with memory (distance evaluation trellis) maximizes the output of the second one (ρ'^2) for all $\mathbf{d}^{(1)} \neq \mathbf{d}^{(2)}$.

B. Initial and final state of distance evaluation trellis

We will assume that the two data messages are identical for $n < 0$, i.e. $d_n^{(1)} = d_n^{(2)}$, $n < 0$. Data symbols $d_0^{(1)}$ and $d_0^{(2)}$ are the first that differ and index $n = 0$ corresponds to the first modulator trellis path split. We also realize that

[5] We can assign the index $m = 0$ to this point without any loss of generality.

[6] Raised Cosine impulse with the length $2T_S$.

[7] The discrete function $(d_n, \sigma_n) \mapsto \mathbf{q}_n$ forms a discrete part of the modulator.

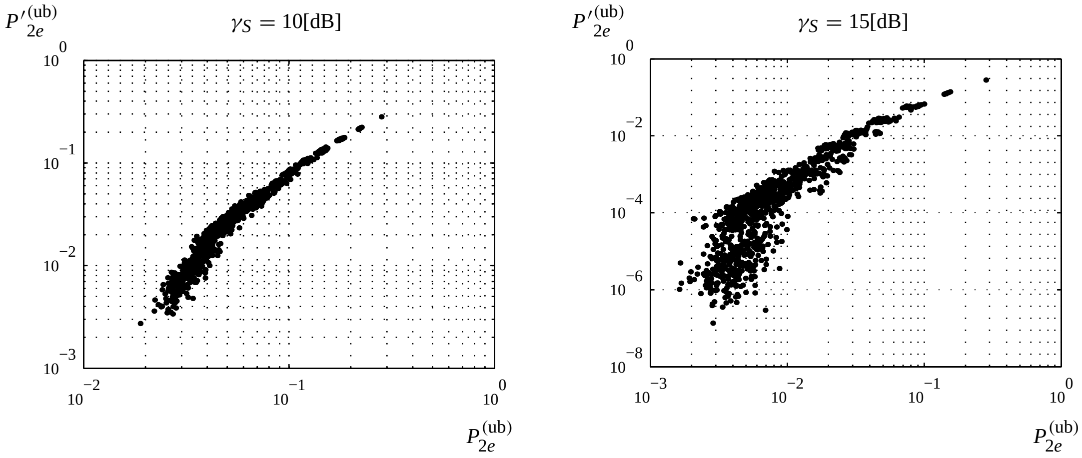

Fig. 1. Relation between $P'^{(\mathrm{ub})}_{2e}$ and $P^{(\mathrm{ub})}_{2e}$ for the example case III-C.

the initial state of the distance evaluation trellis is always (for any particular data) given by (see section III-B)

$$\boldsymbol{\theta}_i(0) = [0, 0, \ldots 0]^T \tag{32}$$

at the point of the path split.

A situation is much more interesting at the point in the modulator trellis where the two paths merge again. When the paths merge again at $n = K$, where K is the length of the free path, then $\Delta \mathbf{q}_n = \mathbf{0}$ for all $n \geq K$. However the behavior of the distance evaluation trellis is very different. The distance evaluation trellis reaches its final state (after some delay given by the memory of the evaluation procedure) after the merge in the modulator trellis at $n = K$. A particular value of the final state depends on the distance evaluation trellis and on the particular history of channel phase symbol *differences* $\Delta \mathbf{q}_n$. The number of possible final states is $M_\eta^{N_t}$ which can be easily deduced from the distance evaluation state transition equation. All possible final states are

$$\boldsymbol{\theta}_i(m) = [\psi_i(m), 0, 0, \ldots, 0]^T, \tag{33}$$

$i = 1, \ldots, N_t$.

One of the possible final states is *equivalent* to the initial state $\boldsymbol{\Theta}(m) = \mathbf{0}$. We realize that the squared distance increment is *zero* $\Delta\rho'^2(m) = 0$ *if and only if* the distance evaluation state is $\boldsymbol{\Theta}(m) = \mathbf{0}$ and $\Delta \mathbf{q}_m = \mathbf{0}$. This means that there are some paths (with the final state $\boldsymbol{\Theta}(m) = \mathbf{0}$) in the distance evaluation trellis that *stop to increase* the accumulated squared distance when they reach the final state. There are also paths (with the final state $\boldsymbol{\Theta}(m) \neq \mathbf{0}$) that *continue to increase* the squared distance even when they reach the final state. The paths with the final state $\boldsymbol{\Theta}(m) = \mathbf{0}$ will be referred as *critical distance evaluation trellis paths*. These paths affect the minimum free distance of the modulated signal. All other paths continue to increase the distance even after the merge in the modulator trellis and therefore cannot influence the minimum free distance.

It is also useful to observe that in order to get into the final state $\boldsymbol{\Theta}(m)$ the last $L-1$ channel symbol differences must be zero $\Delta \mathbf{q}_n = \mathbf{0}$ for $n = (m-L+1), \ldots, (m-1)$. Thus only first $m - L + 1$ differences $\{\Delta \mathbf{q}_n\}_{n=0}^{m-L}$ needs to be explored for reaching the final state at m.

C. Modulator trellis design rules

Here, we formulate modulator trellis design rules based on the previous conclusions on the behavior of the distance evaluation trellis (especially its final states). The main concern is the critical distance evaluation trellis path. The design rules are:

1. We analyze the distance evaluation trellis and find the shortest free critical paths. The length of the shortest one is denoted by ξ_0. The longer ones are denoted by $\xi_n = \xi_0 + n,\ n \in \mathbb{N}$.
2. We identify the set of all sequences of the channel phase symbols differences

$$\mathbb{A}_\ell = \{\Delta\vec{\mathbf{q}} = [\Delta \mathbf{q}_0, \Delta \mathbf{q}_1, \ldots, \Delta \mathbf{q}_{\xi_\ell - L}]^{(p)}\}_{p=1}^{M_{\xi_\ell}} \tag{34}$$

that *induce the critical path* in the distance evaluation trellis. Especially important is the set corresponding to the shortest free path $\mathbb{A}_0$. We also find corresponding squared distances for all critical paths $\rho'^2_{c,\ell}(\Delta\vec{\mathbf{q}}) = \rho'^2$ for all $\Delta\vec{\mathbf{q}} \in \mathbb{A}_\ell$ and define

$$\rho'^2_{c\,\min,\ell} = \min_{\Delta\vec{\mathbf{q}} \in \mathbb{A}_\ell} \rho'^2_{c,\ell}(\Delta\vec{\mathbf{q}}). \tag{35}$$

3. We design the modulator trellis in such a way that we *avoid* any sequence $\Delta\vec{\mathbf{q}} \in \mathbb{A}_0$ for any pair of possible paths in the modulator trellis that splits at the common point and merges after $\xi_0 - L + 1$ transitions.
4. The pairs of free paths having the length $\xi_1 - L + 1$ in the modulator trellis will be assigned to such channel symbols that $\Delta\vec{\mathbf{q}} \in \mathbb{A}_1$ and correspond to the highest possible $\rho'^2_{c,1}(\Delta\vec{\mathbf{q}})$. This guarantees maximization of

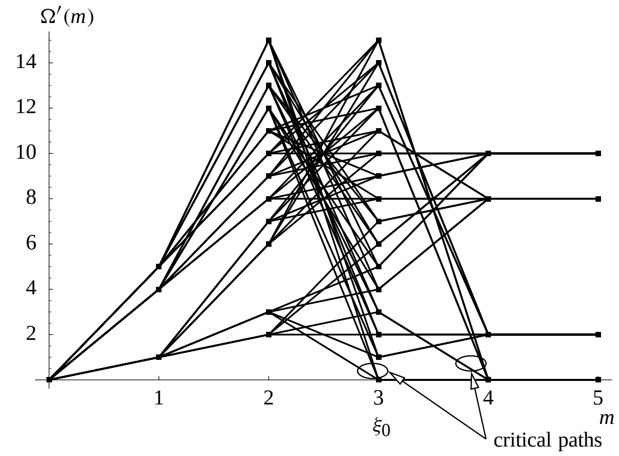

Fig. 2. The distance evaluation trellis with reduced set of the states for the example III-C.

$\rho'^2_{c\,\min,1}(\mathbb{U}_1)$ where the minimum free squared distance over all used $\Delta\vec{\mathbf{q}} \in \mathbb{U}_\ell$ is

$$\rho'^2_{c\,\min,\ell}(\mathbb{U}_\ell) = \min_{\Delta\vec{\mathbf{q}}\in\mathbb{U}_\ell} \rho'^2_{c,\ell}(\Delta\vec{\mathbf{q}}) \tag{36}$$

and the set of all *used* difference sequences is defined as

$$\mathbb{U}_\ell = \{\Delta\vec{\mathbf{q}},\ \text{used at critical path with}\ \xi_\ell\ \text{length}\} \subset \mathbb{A}_\ell. \tag{37}$$

The expression $\rho'^2_{c\,\min,1}(\mathbb{U}_1)$ can be greater then $\rho'^2_{c\,\min,1}$ since there can be some $\Delta\vec{\mathbf{q}} \in \mathbb{A}_1$ not actually used in the modulator trellis. On the other side, the particular situation for given modulation parameters *generally does not* guarantee that $\rho'^2_{c\,\min,\ell_2}$ would be greater than $\rho'^2_{c\,\min,\ell_1}(\mathbb{U}_{\ell_1})$ for $\ell_2 > \ell_1$. A careful observation of this behavior is needed. However the situation for high difference values $\ell_2 - \ell_1$ usually satisfies the condition.

D. Example case

Here, we will continue with the example case III-C. The distance evaluation trellis free paths are shown on Fig. 2 and 3. The reduced states are indexed by a new variable $\Omega'(m)$ which is get from the reduced states $\boldsymbol{\theta}'_i(m)$ as

$$\Omega'(m) = [8, 4]\boldsymbol{\theta}'_1(m) + [2, 1]\boldsymbol{\theta}'_2(m). \tag{38}$$

We use similar indexation by $\Omega(m)$ for ordinary states $\boldsymbol{\theta}_i(m)$. It can be clearly seen that the shortest free critical path has the length $\xi_0 = 3$.

The optimum code search in this simple case could be done by the following way.

1. We determine the sequences $\mathbb{A}_0$. An analysis of the distance evaluation trellis reveals that $M_{\xi_0} = 24$ and the set $\mathbb{A}_0$ is

$$\mathbb{A}_0 = \bigcup_i \mathbb{A}_{0,i}$$

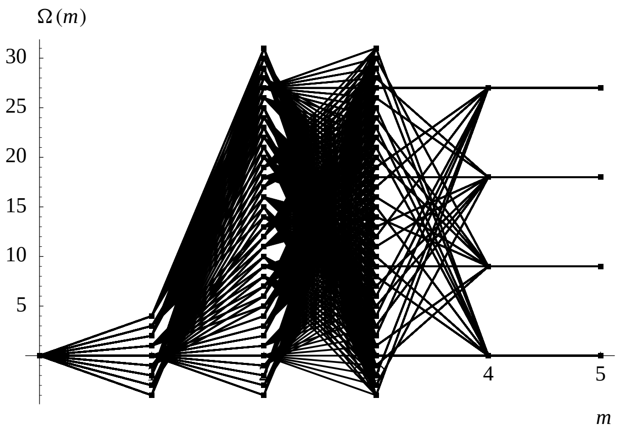

Fig. 3. The distance evaluation trellis for the example III-C.

where

$$\mathbb{A}_{0,1} = \left\{[\pm 2, \pm 2]^T, [\pm 2, \pm 2]^T\right\}, \tag{39}$$

$$\mathbb{A}_{0,2} = \left\{\left\{[\pm 2, 0]^T, [\pm 2, 0]^T\right\}, \left\{[0, \pm 2]^T, [0, \pm 2]^T\right\}\right\}. \tag{40}$$

Mean squared distances for these sets are

$$\rho'^2(\mathbb{A}_{0,1}) \cong 4.0, \tag{41}$$

$$\rho'^2(\mathbb{A}_{0,2}) \cong 2.0. \tag{42}$$

2. Then we find sequences $\mathbb{U}_1 = \bigcup_i \mathbb{U}_{1,i}$ with the largest accumulated squared distances. In this example, the subsets with largest distances are

$$\mathbb{U}_{1,1} = \left\{[\pm 2, \pm 2]^T, [0, 0]^T, [\pm 2, \pm 2]^T\right\}, \tag{43}$$

$$\begin{aligned}\mathbb{U}_{1,2} = \Big\{&\left\{[\pm 2, 0]^T, [0, \pm 2]^T, [\pm 2, \pm 2]^T\right\},\\ &\left\{[0, \pm 2]^T, [\pm 2, 0]^T, [\pm 2, \pm 2]^T\right\},\\ &\left\{[\pm 2, \pm 2]^T, [\pm 2, 0]^T, [0, \pm 2]^T\right\},\\ &\left\{[\pm 2, \pm 2]^T, [0, \pm 2]^T, [\pm 2, 0]^T\right\}\Big\},\end{aligned} \tag{44}$$

$$\vdots$$

$$\mathbb{U}_{1,i} = \cdots,\quad i > 2. \tag{45}$$

Mean squared distances for these sets are

$$\rho'^2(\mathbb{U}_{1,1}) \cong 8.0, \tag{46}$$

$$\rho'^2(\mathbb{U}_{1,2}) \cong 6.0, \tag{47}$$

$$\vdots$$

$$\rho'^2(\mathbb{U}_{1,i}) < \rho'^2(\mathbb{U}_{1,2}),\quad i > 2. \tag{48}$$

3. Then the trellis is set up in such a way that for $m = 0$ the diverting paths, for $m = 1$ the disjunct paths and

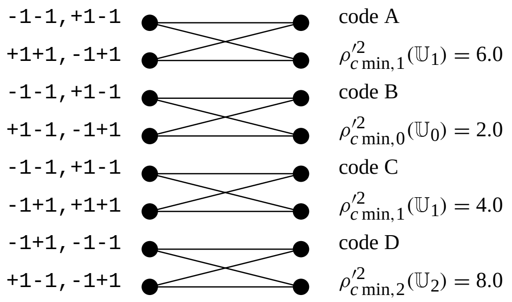

Fig. 4. An example of 2-space 2-state ($M_\sigma = 2$) modulator trellis for binary data ($M_d = 2$) for the example III-C. The vectors at the left hand side are the channel phase symbol vectors. The first is used for the upper and the second for the lower branch splitting at the given state.

for $m = 2$ the merging paths are assigned such channel symbols that their difference falls into subset $\mathbb{U}_{1,i}$ with largest possible distance.

4. Next level of the optimization procedure is the choice of the used paths set avoiding the critical paths even on the lengths $\xi_\ell,\ \ell \geq 1$. However the methodology for this type of design needs more research effort and currently relies on the ad-hoc approach.

Fig. 4 shows a simple 2-space 2-state ($M_\sigma = 2$) modulator trellis for binary data ($M_d = 2$). In the case of the 2-state code we are however very *restricted* in possible mappings of channel symbols onto the code trellis. Therefore we, in fact, cannot use the best $\mathbb{U}_{1,1}$. In order to have the disjunct paths of the $\Delta\mathbf{q}_n = [0, 0]^T$ type, both diverting and merging paths would be from the same set and therefore from the set $\mathbb{A}_0$. In order to utilize $\mathbb{U}_{1,1}$ the coder trellis would have to have *more* than 2 states. We must therefore use the set $\mathbb{U}_{1,2}$. Diverting paths will be from $\Delta\mathbf{q}_n = [\pm 2, 0]^T$, disjunct paths from $\Delta\mathbf{q}_n = [0, \pm 2]^T$ and merging paths from $\Delta\mathbf{q}_n = [\pm 2, \pm 2]^T$.

There are shown several different codes among which the code A is the one with the best possible (for 2-state code) set $\mathbb{U}_{1,i}$. For comparison purposes, Fig. 4 shows also other non-optimal codes. The code B has the shortest used critical path length equal to the minimal one—i.e. $\xi_0 = 3$. Code C has the shortest critical path length $\xi_1 = 4$ however the set $\mathbb{U}_1$ is not the optimal one. There is also shown $\rho'^2_{c\,\min,\ell}(\mathbb{U}_\ell)$ where ℓ corresponds to the shortest used critical path having length ξ_ℓ. The last shown code D is a "hand crafted" design trying to avoid the critical paths even on the length ξ_1. We are rewarded by the largest distance.

E. Open problems

There is a number of open problems that will need to be addressed in the future work. The most important ones are:

- general relation of $\rho'^2_{c\,\min,\ell_2}$ and $\rho'^2_{c\,\min,\ell_1}(\mathbb{U}_{\ell_1})$ for $\ell_2 > \ell_1$,
- general relation between the number of the modulator states M_σ and the availability of sufficiently large set $\mathbb{U}_\ell$ with high $\rho'^2_{c\,\min,\ell}(\mathbb{U}_\ell)$,
- design avoiding critical paths on $\xi_\ell,\ \ell \geq 1$.

V. Conclusions

In this paper, we have presented a general procedure for the space-time constant envelope trellis coded modulation design. The work was concentrated around several key points. The first one was the derivation of the averaged squared distance in the slowly flat fading Rayleigh channel which was shown to correspond to the mean of the error probability logarithm. In the case of the nonlinear modulation this is the only mathematically tractable way. Second major point shown in the paper is the fact that the evaluation of the distance is controlled by two procedures with memory—the discrete part of the modulator and the distance evaluation algorithm. Both can be described by a trellis. The distance evaluation trellis has some special properties related to its initial and final state and also to the distance increments. All these properties were closely investigated. The most important is the concept of the critical distance evaluation trellis path which dominantly determines the overall modulation free distance. The principle of critical distance evaluation path can be generally utilized in any other situation where we design a system with memory while optimizing the objective function itself possessing the memory. In the second part of the paper a set of rules for the modulator trellis code design was suggested. The main attention was focused to the critical path treatment. As an example, a simple 2-space 2-state binary trellis code was designed.

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