Factor Graph Framework for Serially Concatenated Coded CPM with Limiter Phase Discriminator Receiver

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The paper addresses the system with the Serially Concatenated Coded CPM (SCCCPM) and the nonlinear limiter phase discriminator receiver. We develop the FG framework allowing the use of the SPA factor node marginalization rules involving canonical distributions with finite number of parameters. We show that the straightforward application of the traditional constellation space approach is intractable. We also show that the sampled signal space does not allow an application of the parametrized canonical distributions for the variable nodes densities. The solutions we suggest in the paper stands on using the sampled parametric (phase) space of the CPM for all SPA processing. We use the idea of the component modulo mean approximation for the canonical signal phase distributions. The modulo operations are applied only on component mean values. This keeps the update rules simple by avoiding the circular distortion of the Gaussian component densities while still capturing the circular nature of the modulo operations on the signal phase.

I. Introduction

A. Background

The SCCCPM has attracted recently attention as a a scheme combining the excellent error performance with capability of an iterative decoding with the ideal CPM resistance to the nonlinear distortion due to its constant envelope [1]. The suitability of SCCCPM for the iterative decoding is due to the presence of the continuous phase encoder [2] in the CPM having in fact a role of the inner recursive encoder of the concatenated scheme. The Factor Graphs (FG) and the Sum-Product Algorithm (SPA) ([3], [4]) is a general framework able to capture a variety of the signal processing algorithms including the iterative soft information passing turbo decoding in the serially concatenated scenario [5], [6]. The FG/SPA technique can be also used to inherently cope with the parametric channel introducing nuisance channel parameters of various forms (phase, channel transfer, etc.) [7], [8], [9], [10]. One of the major problems arising from the inclusion of continuous valued channel parameters is the fact that the SPA becomes practically difficult to implement. The SPA defines the marginalization update rules for the factor node output messages (see [3], [4]) where the continuous parameters with general densities introduce the necessity to integrate when calculating the factor node marginalization. This is of course very impractical. Several solutions (approximations) solving this problem appeared. They are centered around the idea of using the approximation of the densities by the restricted class of *parameterizable* canonical distributions [11], [12]. The marginalization factor node update thus evaluates only the new set of the density parameters. The excellent resistance of the CPM to the nonlinear distortion also allows *Limiter Phase Discriminator (LPD)* receiver processing where the signal first passes through the limiter and then the instant phase is observed. This approach has a substantial advantage of allowing a very *robust* receiver design not requiring the gain control otherwise needed for proper signal level adjustment at the analog-digital conversion. Strictly speaking, the phase of the signal itself does not provide a sufficient statistic for the CPM modulation in AWGN. However, it is shown in [13] that the mutual information loss is *negligible*.

B. The contribution of the paper

We develop the FG framework allowing the use of the SPA factor node marginalization rules involving canonical distributions with finite number of parameters for the nonlinear *limiter phase discriminator receiver*. We show that the straightforward application of the traditional constellation space approach (as it is used for linear modulations) is intractable. We also show that the sampled signal space does not allow an application of the parametrized canonical distributions for the variable nodes densities. The solutions we suggest in the paper stands on the usage of the sampled parametric (phase) space of the CPM for all FG/SPA processing.

The core is in the efficient representation of the densities passed over over the FG edges between Factor Nodes (FN) and Variable Nodes (VN). These messages must cope with the *circular nature* of the modulo 2π operations of the signal *phase* processing. On the other side, the message update rules must be simple enough. This would cause a problem if we strictly apply all modulo operations on all random variables in the FG. This would lead to *non-Gaussian* densities which are

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difficult to cope with. We show the solution based on *modulo component mean approximation*. This keeps the *circular nature* of the phase modulo operations by applying them on *variable component means*. The Gaussian *tails* are however allowed to *extend* out of the $[0, 2\pi)$ range. It will keep the FG/SPA update rules simple since all the densities keep to be Gaussian ones. The canonical distributions in [11], [12] do not treat the problem of modulo signal phase operations. They rather operate on the sample or the constellation space domains where this problem does not appear. The modulo circular operations appear only for the direct signal phase space operations (i.e. the phase discriminator receiver).

The presented framework for modulo mean canonical distributions is developed as *modular tool* that allows to incorporate the CPM and the phase discriminator receiver processing into the complete FG with serially concatenated coding. This is however outside of the scope of the paper.

II. System Model and Definitions

We consider a message of binary data symbols $\mathbf{b} = [\ldots, b_n, \ldots]^T$ encoded by the outer encoder into the codeword $\mathbf{c} = [\ldots, c_n, \ldots]^T$ passing through the interleaver Π having at its output the interleaved codeword $\mathbf{d} = [\ldots, d_n, \ldots]^T$. This forms an input of the CPM modulator. It is modeled using the Continuous Phase Encoder (CPE) with output $M_q$ary symbols $q_n \in \{0, \ldots, M_q - 1\}$ and the Nonlinear Memoryless Modulator (NMM) [2] with output signal $s(t) = \exp(j\phi(t))$, where $\phi(t) = \kappa \sum_n \beta(q_n, t - nT_S)$. The κ is the modulation index, β is the nonlinear phase function of the NMM and T_S is the symbol period. The channel transfer $h = \exp(j\varphi)$ is assumed to be with the random phase rotation φ having uniform distribution. The AWGN is $w(t)$. The received signal $x(t)$ passes through the *nonlinear* LPD with the output $\theta = \measuredangle(x)$. See Fig. 1.

III. FG Framework for SCCCPM with LPD

A. Multidimensional Constellation Space

The first option for developing the FG representation of the system is the one using the Multidimensional (mD) Constellation Space (CS) (Fig. 2). The CS is *multidimensional* due to the *nonlinearity* of the CPM. The CS approach is the dominant one for linear modulations (like PSK, QAM) where, due to the linearity, the space dimension is equal to one. Thus it is very simple situation to handle with unlike for our case of CPM.

The CS representation of the transmitted signal, the channel transfer, and the received signal are mD vectors $\mathbf{s}_n, \mathbf{h}_n, \mathbf{x}_n$ for nth symbol q_n. The messages associated with VN q_n and $\mathbf{s}_n$ are the discrete probability mass functions. Those are easy to handle within the SPA. The messages associated with $\mathbf{u}_n$,$\mathbf{x}_n$ are the Mixture Gaussian (MG) ones [11], i.e. the weighted superposition of component Gaussian densities. Those are not particularly easy to implement but still can be practically handled using various approximations with good results (see [11]) in the SPA. However the major problem of the mDCS approach is the FN corresponding to the LPD. There is no direct relation between the coefficients of the CS expansion and the phase parameter θ, unless we reconstruct the signal in time using the basis and the coefficients. This creates a complicated input-output relation and makes the problem *intractable*.

B. Sampled Signal Space

The Sampled Signal Space (SaSS) approach (Fig. 3) tries to solve the problem of LPD FN by performing all processing directly on complex samples. The signal must be sampled $N_S > 1$ times per one q_n symbol. We denote the kth sample $s_{n,k}$ and similarly for other variables. This makes the definition of the LPD FN relatively straightforward $p(\theta_{n,k}|x_{n,k}) = \delta(\theta_{n,k} - \measuredangle(x_{n,k}))$. A formulation of other factors is also easy. The CPM signal definition directly dictates $p(s_{n,k}|q_n)$. The channel transfer factor is $p(u_{n,k}|h_{n,k}, s_{n,k}) = \delta(u_{n,k} - h_{n,k}s_{n,k})$. The AWGN FN is easily defined by the conditionally Gaussian PDF $p(x_{n,k}|u_{n,k})$. However the easy situation with the definitions of the factors is badly compensated by the fact that messages (PDF) at VN $s_{n,k}$ and $u_{n,k}$ has a complicated form of the continuous random variable defined at the circle in the complex plane. This is given by the form of the signal $s(t) = \exp(j\phi(t))$ and $u(t) = s(t)h(t) = \exp(j\alpha(t))$. At initial iteration, it can be regarded as a uniform phase PDF, but with iterations proceeding it becomes a complicated function with number of local extremes around the circle in complex plane (see Fig. 3). The situation becomes even worse at VN $x_{n,k}$ where this is further convolved with *complex* Gaussian PDF of the AWGN FN. Those PDF cannot be easily described by the finite set of parameters and therefore this approach is very difficult for implementation.

C. Sampled Phase Space

The solution of the all above stated problems is the Sampled Phase Space (SaPS) (Fig. 4). This approach is motivated by the Information Waveform Manifold (IWM) approach of [13] where the IWM is used in a more general setup of CPM in MIMO channel. The idea we use here stands on the step where we move the whole processing into the parametric space of the CPM signal, i.e. its *phase* parametric space. This is the space spanned by the ϕ phase of the signal $s(t) = \exp(j\phi(t))$ and similarly for $u(t) = s(t)h(t) = \exp(j\alpha)$.

The factors $p(\phi_{n,k}|q_n)$ are directly given by the functions β and form the discrete probability mass functions of VN $\phi_{n,k}$. The factor of the channel transfer FN is $p(\alpha_{n,k}|\varphi_{n,k}, \phi_{n,k}) = \delta(\alpha_{n,k} - (\varphi_{n,k} + \phi_{n,k}))$. The densities at VN $\alpha_{n,k}$ start at initial iteration as the uniform ones and with the iterations proceeding they become of the mixture Gaussian type. The AWGN channel FN factor $p(\theta_{n,k}|\alpha_{n,k})$, where $\theta_{n,k} = \measuredangle(\exp(j\alpha_{n,k}) + w_{n,k})$, can be *approximated* by the *Gaussian* density for moderate signal-to-noise ratios. This approximation effectively transfers the case of CPM with LPD demodulation treated in SaPS into the system similar to the traditional case of linear modulation in AWGN channel. We can use the message update rules similar to [11] (the approximations) applicable to the mixture Gaussian densities. This means that only a finite set

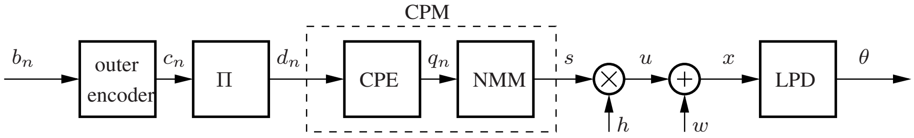

Figure 1. The system model.

of density parameters is calculated at each update. The update rules will use the *modulo component mean approximation*. The FN update will not involve any integration. Moreover, all mixture Gaussian densities are *real-valued* unlike for the SaSS case.

IV. Message types

The messages passed in the FG (Fig. 4) can be modeled by two basic message types. The first is a traditional discrete probability mass function for the integer valued random variable. The second message type must be capable of describing *real* mixture Gaussian random variable (i.e. superposition of *real* valued discrete and *real* Gaussian random variable).

A. Discrete message type

The discrete message type is associated with VN q_n. The backward message (a priori probability mass function) is $\mathcal{M}_{\mathrm{B}}\{q_n\} = \{p(q^{(i)})\}_{i=0}^{M_q-1}$ and the forward message (a posteriori probability for a given observation) is $\mathcal{M}_{\mathrm{F}}\{q_n\} = \{p(\boldsymbol{\theta}_n^0|q^{(i)})\}_{i=0}^{M_q-1}$. The decision measure is $\mathcal{M}\{q_n\} = \mathcal{M}_{\mathrm{F}}\{q_n\}\mathcal{M}_{\mathrm{B}}\{q_n\}$. A generic message, that can become any of $\mathcal{M}\{q_n\}, \mathcal{M}_{\mathrm{F}}\{q_n\}, \mathcal{M}_{\mathrm{B}}\{q_n\}$, will be denoted by $\mu\{q_n\}$.

B. Mixture Gaussian modulo mean message type

All remaining messages can be modeled/approximated by Mixture Gaussian Modulo Mean (MGMM) messages. Here we extend the ideas similar to [11], [12] for the case of variables having interpretation of the *phase of the signal*. All signal processing operations are modulo 2π operations. A straightforward application of the principles used in [11], [12] on the mixture Gaussian densities defined over the unity circle (to reflect $\mod_{2\pi}$ phase operations) does not lead to the tractable solution. The circularly overlapping tails of individual Gaussian PDFs cause the loss of the simplicity of the manipulation with Gaussian densities as we are used to for real valued *linear* axis. We propose a solution combining the simplicity of the manipulation of the Gaussian messages with the attributes reflecting the circular nature of the $\mathrm{mod}_{2\pi}$ phase operations. It a MGMM $\mathrm{mod}_{2\pi}$ solution.

The MGMM $\mathrm{mod}2\pi$ messages are formed by parametric class of the densities

$$p(x) = \sum_{i=1}^{M_x} p(x^{(i)}) p_{\mathcal{N}(\sigma_x^2)}(x - x^{(i)}) \tag{1}$$

where $p_{\mathcal{N}(\sigma_x^2)}(x)$ is the PDF of *zero-mean*, σ_x^2 variance *real-valued* Gaussian random variable, and $p(x^{(i)})$ is the probability

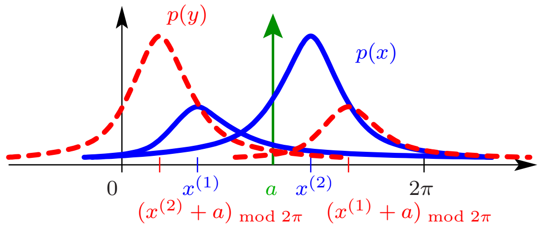

Figure 5. An example of the modulo mean equivalence.

of the ith component. The *generic* messages are

$$\mu\{x\} = \left\{ \{p(x^{(i)})\}_{i=1}^{M_x}, \{x^{(i)}\}_{i=1}^{M_x}, \sigma_x^2 \right\}, \quad x^{(i)} \in [0, 2\pi). \tag{2}$$

All operations with MGMM type variables are performed with $\mathrm{mod}\,2\pi$ restriction *only* applied on the set of *component mean values* $\{x_i\}_i$. The tails of the component Gaussian densities are allowed to go outside the $[0, 2\pi)$ range. The support of the MGMM PDF is the complete real axis, however the main probability mass lies within the $[0, 2\pi)$ range. This *approximates* the *circular nature* of the true $\mathrm{mod}2\pi$ arithmetics but keeps the composite densities *Gaussian* and therefore *easily tractable*. The resulting equivalent (approximate) density will be called *Modulo Mean Equivalent Density* (MMED).

The MGMM messages can easily model (as a special case) discrete valued variables by simply setting the component variance to zero $\sigma^2 \to 0$. The uniform density (e.g. for initial iteration) can be approximated by setting large σ^2.

C. Example of the modulo mean equivalence

Let us consider a simple example of the operation $y = (x + a) \mod_{2\pi}$. The PDFs are all of the MGMM type and a is a constant. The example demonstrates how the modulo operation is performed only on the component mean values keeping the result as a sum of Gaussian components but still reflecting the nature of the circular operation. See Fig. 5 for the resulting *equivalent* density $p(z)$ (in the sense of the *modulo mean equivalence*).

V. Message update rules

Here, we derive the message update rules for the FNs and VNs used in SaPS processing FG/SPA. All messages are assumed to be the MGMM ones (1). We concentrate only on those particular FN and VN relevant to the CPM and the phase discriminator processing used in our system model (Fig. 4).

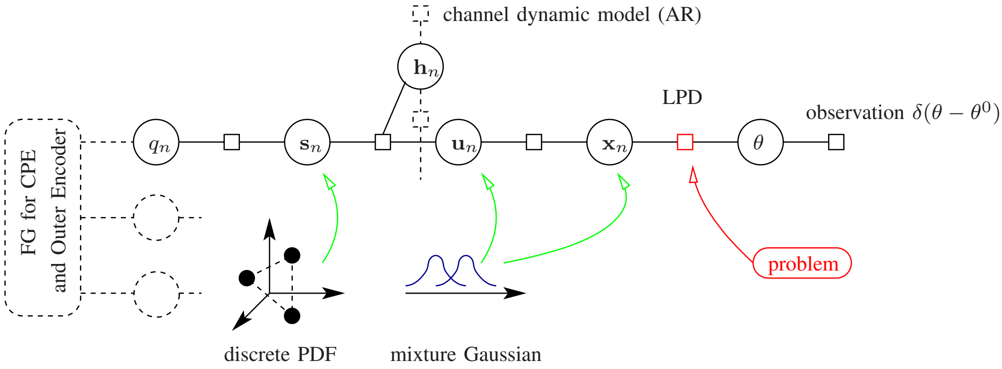

Figure 2. The Multidimensional Constellation Space FG model.

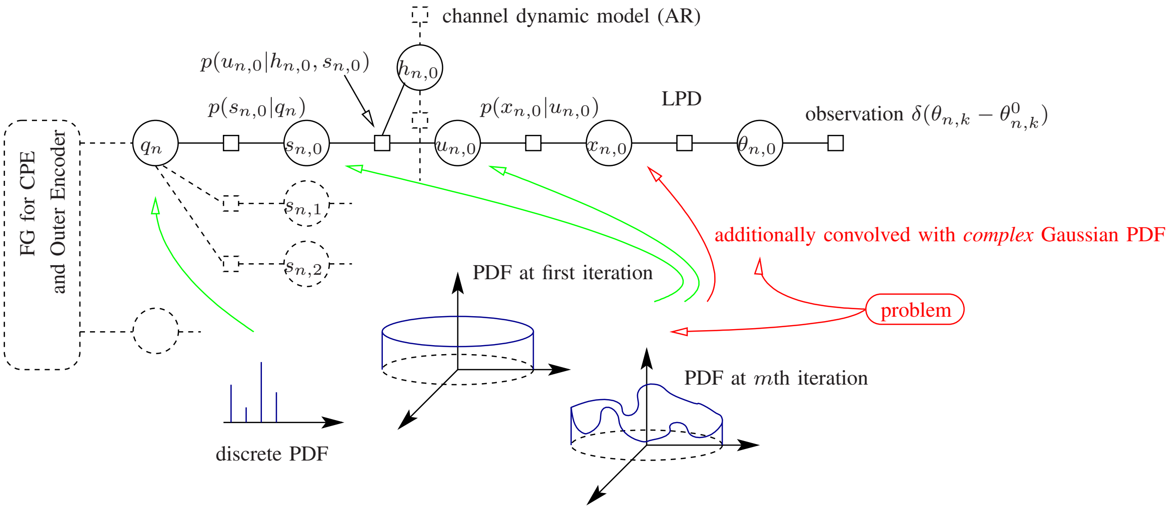

Figure 3. The Sampled Signal Space FG model.

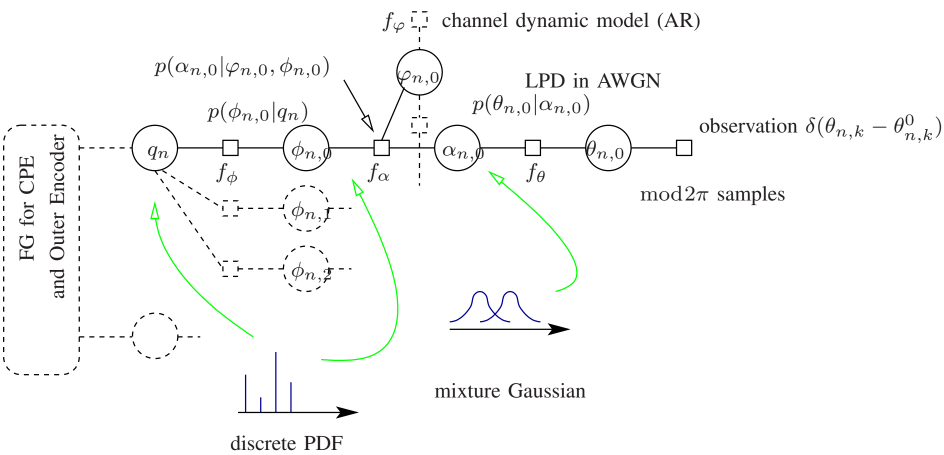

Figure 4. The Sampled Phase Space FG model.

A. (q, w, x) FN $x = (q + w) \bmod {}_{2\pi}$ *for MGMM messages*

We assume FN performing an operation $x = (q+w) \bmod {}_{2\pi}$ and having 3 edges (q, w, x) (Fig. 6). The associated FN PDF factor is

$$p(x|q, w) = \delta(x - (q + w) \bmod {}_{2\pi}). \tag{3}$$

The message $\mu\{w\}$ is assumed to have only a *single component*, i.e. $M_w = 1$. The message $\mu\{q\}$ is assumed to be a *discrete* one, i.e. $\sigma_q^2 \to 0$. The number of components in $\mu\{x\}$ and $\mu\{q\}$ is equal $M_x = M_q$. This FN is applicable for f_α in Fig. 4. The sampled phase values $\phi_{n,k}$ are discrete ones with the set of values given by the particular CPM phase function $\beta(\boldsymbol{.})$. The channel transfer phase $\varphi_{n,k}$ is on the other side a single value with Gaussian perturbations.

1) $\mu\{x\}$ message: The FN message to the x VN has generally the PDF

$$\begin{aligned} p(x) &= \sum_i \int p(x|q^{(i)}, w)p(q^{(i)})p(w)\, dw \\ &= \sum_i p(q^{(i)}) \int \delta(x - (q^{(i)} + w) \bmod {}_{2\pi})p(w)\, dw. \end{aligned} \tag{4}$$

For $\sigma_w^2 \ll 2\pi$, we can approximate

$$\delta(x - (q + w) \bmod 2\pi) \approx \delta(x - (q^{(i)} + w^{(1)}) \bmod {}_{2\pi} - \Delta w) \tag{5}$$

where $w = w^{(1)} + \Delta w$, $w^{(1)}$ being the mean value. Then the update rule for $\mu\{x\}$ message is

$$p(x^{(i)}) = p(q^{(i)}), \tag{6}$$

$$x^{(i)} = (q^{(i)} + w^{(1)}) \bmod {}_{2\pi}, \tag{7}$$

$$\sigma_x^2 = \sigma_w^2. \tag{8}$$

2) $\mu\{w\}$ message: The FN message to the w VN has generally the PDF

$$\begin{aligned} p(w) &= \sum_i \int p(x|q^{(i)}, w)p(q^{(i)})p(x)\, dx \\ &= \sum_i p(q^{(i)}) \int \delta(x - (q^{(i)} + w) \bmod {}_{2\pi})p(x)\, dx \\ &= \sum_i \sum_j p(q^{(i)})p(x^{(j)})p_{\mathcal{N}(\sigma_x^2)}((q^{(i)} + w) \bmod {}_{2\pi} - x^{(j)}). \end{aligned} \tag{9}$$

This is a weighted sum of shifted Gaussian densities. The $\mu\{w\}$ message is assumed to be a single component one. The assumption of a single component $\mu\{w\}$ message requires the *ambiguity* resolution. We adopt a simple *ambiguity resolution strategy* based on selecting the most probable components in the messages $\mu\{q\}$ and $\mu\{x\}$. The update rule is

$$p(w^{(1)}) = 1, \tag{10}$$

$$w^{(1)} = x^{(\arg\max_j p(x^{(j)}))} - q^{(\arg\max_i p(q^{(i)}))}, \tag{11}$$

$$\sigma_w^2 = \sigma_x^2. \tag{12}$$

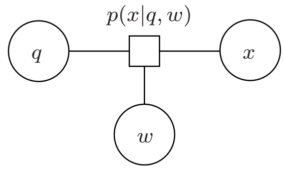

Figure 6. (q, w, x) FN $x = (q + w) \bmod {}_{2\pi}$.

3) $\mu\{q\}$ message: The FN message to the q VN has generally the probabilities of the values

$$\begin{aligned} p(q^{(i)}) &= \int\int p(x|q^{(i)}, w)p(x)p(w)\, dxdw \\ &= \int\int \delta(x - (q^{(i)} + w) \bmod {}_{2\pi})p(x)p(w)\, dxdw \\ &= \sum_j p(x^{(j)}) \int p_{\mathcal{N}(\sigma_x^2)}((q^{(i)} + w) \bmod {}_{2\pi} - x^{(j)})p(w)\, dw. \end{aligned} \tag{13}$$

For $\sigma_w^2 \ll 2\pi$, we can approximate (similarly as for $\mu\{x\}$)

$$\begin{aligned} &p_{\mathcal{N}(\sigma_x^2)}((q^{(i)} + w) \bmod {}_{2\pi} - x^{(j)}) \\ &\quad \approx p_{\mathcal{N}(\sigma_x^2)}(\Delta w + (q^{(i)} + w^{(1)}) \bmod {}_{2\pi} - x^{(j)}) \end{aligned} \tag{14}$$

where $w = w^{(1)} + \Delta w$. This is Gaussian PDF w.r.t. Δw with the mean $a = x^{(j)} - (q^{(i)} + w^{(1)}) \bmod {}_{2\pi}$ and the variance σ_x^2. Similarly, $p(w) = p_{\mathcal{N}(\sigma_w^2)}(w - w^{(1)}) = p_{\mathcal{N}(\sigma_w^2)}(\Delta w)$ has the zero mean and the variance σ_w^2. Then

$$\begin{aligned} p(q^{(i)}) &= \sum_j p(x^{(j)}) \int p_{\mathcal{N}(\sigma_x^2)}(\Delta w - a)p_{\mathcal{N}(\sigma_w^2)}(\Delta w)\, d\Delta w \\ &= \sum_j p(x^{(j)}) \frac{1}{\sqrt{2\pi(\sigma_x^2 + \sigma_w^2)}} e^{-\frac{a^2}{2(\sigma_x^2 + \sigma_w^2)}}. \end{aligned} \tag{15}$$

The update rule for sample probabilities is

$$p(q^{(i)}) = \sum_j p(x^{(j)}) \frac{e^{-\frac{\left(x^{(j)} - (q^{(i)} + w^{(1)}) \bmod {}_{2\pi}\right)^2}{2(\sigma_x^2 + \sigma_w^2)}}}{\sqrt{2\pi(\sigma_x^2 + \sigma_w^2)}}. \tag{16}$$

The set of values $\{q^{(i)}\}_i$ is defined a priori. The variance for the discrete message type is $\sigma_q^2 \to 0$.

B. (x, y) FN $y = (x + w) \bmod {}_{2\pi}$ *for MGMM messages*

We assume FN with 2 edges (Fig. 7). The x edge has multicomponent MGMM messages $\mu\{x\}$. The y edge is assumed to be connected to the *observation* VN. Therefore its input message is a 1-component MGMM message $\mu\{y\}$ with *Dirac delta PDF*. The associated operation is $y = (x + w) \bmod {}_{2\pi}$ where w is zero-mean Gaussian noise with the variance σ_w^2. In the sense of the *equivalent* modulo mean messages, it can be replaced by $y = x + w$ since w is the zero-mean variable and x and y are having component means in the range $[0, 2\pi)$ anyway.

The factor is then simply

$$p(y|x) = p_{\mathcal{N}(\sigma_w^2)}(y - x). \tag{17}$$

This FN is applicable to f_θ in Fig. 4.

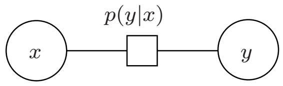

Figure 7. (x, y) FN $y = (x + w) \bmod {}_{2\pi}$.

1) $\mu\{y\}$ message: The FN message to y VN has PDF

$$
\begin{aligned}
p(y) &= \int p_{\mathcal{N}(\sigma_w^2)}(y - x)p(x)\,dx \\
&= \sum_i p(x^{(i)}) \int p_{\mathcal{N}(\sigma_w^2)}(y - x)p_{\mathcal{N}(\sigma_x^2)}(x - x^{(i)})\,dx \\
&= \sum_i p(x^{(i)}) \frac{1}{\sqrt{2\pi(\sigma_x^2 + \sigma_w^2)}} e^{-\frac{(y - x^{(i)})^2}{2(\sigma_x^2 + \sigma_w^2)}} \\
&= \sum_i p(x^{(i)}) p_{\mathcal{N}(\sigma_x^2 + \sigma_w^2)}(y - x^{(i)}). \qquad (18)
\end{aligned}
$$

The update rules are

$$p(y^{(i)}) = p(x^{(i)}), \qquad (19)$$

$$y^{(i)} = x^{(i)}, \qquad (20)$$

$$\sigma_y^2 = \sigma_x^2 + \sigma_w^2. \qquad (21)$$

2) $\mu\{x\}$ message: The $\mu\{y\}$ message is assumed to stem from the *observation* VN, thus $p(y) = \delta(y - y^{(1)})$. Then

$$
\begin{aligned}
p(x) &= \int p(y|x)p(y)\,dy \\
&= \int p_{\mathcal{N}(\sigma_w^2)}(y - x)\delta(y - y^{(1)})\,dy. \qquad (22)
\end{aligned}
$$

The update rules are

$$p(x^{(i)}) = \begin{cases} 1, & i = 1 \\ 0, & \text{otherwise} \end{cases}, \qquad (23)$$

$$x^{(i)} = y^{(i)}, \qquad (24)$$

$$\sigma_x^2 = 0. \qquad (25)$$

C. (w_1, w_2, w_3)VN for 1-component MGMM messages

We assume VN with 3 connected edges with the messages $\mu\{w_1\}, \mu\{w_2\}, \mu\{w_3\}$. All messages are assumed to be 1-component MGMM ones. This is applicable to the VN $\varphi_{n,k}$ in Fig. 4.

The general update rule for PDF associated with w_3 is (the VN is completely symmetric)

$$p(w_3) = A p_{\mathcal{N}(\sigma_{w_1}^2)}(w_3 - w_1^{(1)}) p_{\mathcal{N}(\sigma_{w_2}^2)}(w_3 - w_2^{(1)}) \qquad (26)$$

where A is a normalization scaling factor set to have $\int p(w_3)\,dw_3 = 1$. The PDF $p(w_3)$ can be easily interpreted as a cut over the line $w_1 = w_2$ in a 2-dimensional Gaussian PDF with the mean $[w_1^{(1)}, w_2^{(2)}]$ and variances in the first and the second dimension $\sigma_{w_1}^2, \sigma_{w_2}^2$ respectively. The resulting PDF is again Gaussian and has the mean $(w_1^{(1)}\sigma_{w_2}^2 + w_2^{(1)}\sigma_{w_1}^2)/(\sigma_{w_2}^2 + \sigma_{w_1}^2)$ and the variance $(\sigma_{w_2}^2\sigma_{w_1}^2)/(\sigma_{w_2}^2 + \sigma_{w_1}^2)$. The update equation is then

$$w_3^{(1)} = \frac{w_1^{(1)}\sigma_{w_2}^2 + w_2^{(1)}\sigma_{w_1}^2}{\sigma_{w_2}^2 + \sigma_{w_1}^2}, \qquad (27)$$

$$\sigma_{w_3}^2 = \frac{\sigma_{w_2}^2\sigma_{w_1}^2}{\sigma_{w_2}^2 + \sigma_{w_1}^2}. \qquad (28)$$

VI. CONCLUSIONS

We have developed a framework for FG/SPA algorithm for the *CPM* class of modulations with the *phase discriminator receiver*. We have found canonical distributions for mixture discrete/continuous variables that allow efficient construction of the messages and the corresponding FN and VN update rules. The idea heavily stands on the usage of the *component modulo mean* approximation of the true distributions. It overcomes the problem of the circular modulo 2π operations by applying the modulo operation only on the variable means while leaving the Gaussian tails to extend out of the $[0, 2\pi)$ range in a linear manner. This keeps the update rules simple (due to the Gaussian variables not distorted by the circular operations). The presented framework allows the incorporation of the CPM modulation and the phase discriminator receiver processing as a part of overall FG comprising also the serially concatenated code. The evaluation of this overall system is outside of the scope of this paper.

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