PART II

TECHNICAL PAPERS
RECIPROCITY IN ELECTROMAGNETIC, MECHANICAL, ACOUSTICAL, AND INTERCONNECTED SYSTEMS*

BY

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Summary—Recent criticism by Carson of the statement of the reciprocal relations in a radio communication system given by Sommerfeld is supported by a simple example showing the incorrectness of this statement.

Carson’s proof of the extension of Rayleigh’s reciprocity theorem to a general electromagnetic system was limited to \( \mu = 1 \) and to sources consisting of ponderomotive forces on the electricity. A new proof is given under more general conditions, \( \varepsilon, \mu, \sigma \) being merely restricted to be scalars and the impressed forces are of the electric type introduced by Heaviside and Abraham. These may be regarded as impressed charges and currents, including ether displacement current. The theorem is finally stated in terms of volume and surface integrals, and thus combines the viewpoints of both Lorentz and Carson.

The reciprocity relations in a mechanical system are reviewed.

The interconnection of electrical and mechanical systems is next considered and a “transduction coefficient” is defined. This concept is useful in formulating mechanical problems in electric-circuit form. An example of symmetrical transduction (copper coil in steady magnetic field) is given. The subject of units is taken up and it is proposed that in order to bring the mechanical quantities into agreement with the electrical ones when the latter are expressed in “practical” units, the mechanical quantities be expressed in “mechanical-volts,” amperes, etc., i.e., in “practical-electric units of the mechanical quantities.” A table for converting the principal mechanical quantities from c.g.s. to practical-electrical units is given.

Reciprocity is shown to exist in interconnected electro-mechanical systems with reversible transduction.

The equations of sound propagation in a gas are developed, regarding the velocity and excess pressure as the fundamental quantities. An acoustical reciprocity theorem involving these quantities is then proved.

Interconnections of mechanical and acoustical systems are then discussed and reciprocal relations are shown to exist in the composite system. Such relations are also valid for a system comprising electrical, mechanical and acoustical systems in series connection.

The reciprocity relations in a reversible electrophone are applied in a method of determining the frequency characteristic of an electrophone or microphone. This is called the “method of three electrophones.” The overall transmission curves of the devices taken two at a time are measured and from these data the frequency characteristic of any one of them can be calculated. Only one of the electrophones is required to be reversible.

At this meeting there has been presented a criticism by Mr. J. R. Carson of the Sommerfeld-Pfrang reciprocity theorem for radio communication, which had been reviewed by me.\(^1\) Mr. Carson seems entirely justified in emphasizing the superior generality and fundamental character of his generalization of Rayleigh's theorem, and in challenging the scope and validity of the Sommerfeld-Pfrang theorem in the form stated by Sommerfeld. The incorrectness of the Sommerfeld statement in terms of radiated power and received electric force in the case of an antenna which does not act as a dipol may be illustrated by a simple example, privately communicated by Mr. Lester Hochgraf.

Consider two equal antennas \(A\) and \(B\) some distance apart, oriented as shown in Fig. 1, and excited at such a frequency that there are complete positive and negative current loops. These antennas radiate equal power. Sommerfeld's theorem states that the “received field-strength” (empfangene Feldstärke) at \(B\) due to the radiation of a certain average power from \(A\) will be equal to that at \(A\) with radiation of the same power from \(B\). It will be seen, however, that the electric force produced at \(B\) by \(A\) is finite, whereas that produced at \(A\) by \(B\) is zero because \(A\) is in the equatorial plane of \(B\) and in this plane the radiation vanishes (at large distance).\(^2\) This reductio ad absurdum seems sufficiently convincing and illustrates in a simple way the purport of Mr. Carson's criticism.

In passing it may be pointed out that the application of the theorem which was made in my review (loc. cit.) to the derivation of the transmitting properties of certain antennas over imperfect earth from their calculated receiving properties is in no way affected by this criticism. The only use to which the theorem was put there was to indicate the reciprocity existing between dipols or current elements, an application for which it is perfectly suitable. The theoretical conclusions of the

\(^1\) Proc. I. R. E., 16, 513; April, 1928.
\(^2\) Cf. Fig. 3, Ballantine: Proc. I. R. E., 12, 838; December, 1924.
review concerning aircraft communication have since been verified experimentally.

2. Carson's Proof of Theorem II. The generalization of Rayleigh's theorem to extended electromagnetic systems given by Carson (Theorem II) is of considerable importance. Carson points out that his proof is restricted to a medium in which \( \mu = 1 \). We may also note that his impressed forces are ponderomotive forces on the electricity and do not act to produce a displacement current in the ether.

The \( \mu = 1 \) restriction resulted from Carson's feeling that the current \( u \) should be a linear function of \( E \). This does not appear to be necessary, however, and the restriction may be removed by adding to \( G \) a term \( 1/\lambda \cdot \text{curl } M \). The justification for this step follows from the fact that the added term, combined with \( u \), vanishes when integrated over all space.

3. Proof of the Reciprocity Relations in an Electromagnetic System. An alternative proof of slightly more general reciprocity relations in an extended electromagnetic system may be constructed along the following lines.

In the first place all the constitutive quantities \( \epsilon \), \( \mu \), \( \sigma \) are allowed to differ from unity and may vary from point to point. In the second place the impressed forces will be regarded as purely electrical forces acting upon the total current; that is, upon the etheric displacement current as well as upon the current due to the motion of electricity. In other words the impressed forces will not be restricted, as in Carson's discussion, to ponderomotive or mechanical forces on the electricity. Generalized impressed forces of this type were included by Heaviside and by Abraham. A force acting only on the electricity can be replaced by an impressed current; a generalized force of the type now postulated, acting also on the ether, is equivalent to impressed electric and ether displacement currents combined with an impressed charge distribution. Impressed forces of the Heaviside-Abraham type may be objected to as artificial. It is true that in certain simple cases, as for example that of a generator winding revolving in a steady magnetic field, the seat of the impressed emf may be definitely localized, but in many other cases such forces are useful as a cloak for our ignorance of the exact mechanism of generation. In still other cases we may have sufficient knowledge of the generating mechanism but may wish to avoid unnecessary discussion by including only the

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terminals of the generator in the system. A shielded vacuum-tube generator would constitute an example of this. In such cases the idea of an impressed charge, with its attendant electric and ether displacement currents is perhaps more useful than the ponderomotive $F$ forces acting directly upon the electricity.

Heaviside also introduced impressed magnetic forces and induction. The utility of these seems really questionable and they will be dispensed with here.

The medium is assumed to be linear and isotropic, that is, the constitutive quantities $\epsilon$, $\mu$ and $\sigma$ are scalars independent of the vectors, and the vectors which they connect are always in the same direction and linear vector functions of one another. No attempt will be made to prove that the reciprocity relations fail in the general case where these quantities are second-order tensors. Sommerfeld states, however, that in the case of propagation in a Faraday medium (e.g., ions in the earth's magnetic field) the reciprocal theorem is no longer valid. In this case the $\epsilon$-tensor has the structure:

$$\epsilon = \begin{vmatrix} \epsilon_{11} - i\alpha & 0 \\ i\alpha & \epsilon_{22} \\ 0 & 0 \end{vmatrix}$$

which is skew-symmetric. Leaving aside intuitive suspicions, so far as I know it has not been definitely shown by Sommerfeld or anyone else that in this case, or in the case where $(\epsilon)$ is symmetric (i.e., $\epsilon_{jk} = \epsilon_{kj}$), or in the still simpler case when all components except $\epsilon_{11}$, $\epsilon_{22}$, $\epsilon_{33}$ are zero, the reciprocal relations fail or cannot be properly formulated.

Denoting the impressed electric force by $E_0$, the field equations for the quasi-stationary state (i.e., $\partial/\partial t = i\omega$) may be written in Gaussian units

$$E_0 = \frac{c}{4\pi\lambda} \text{curl } H - E \quad (1)$$

$$\text{curl } E = - i\omega \mu H/c \quad (2)$$

$$\text{div } \mu H = 0 \quad (3)$$

$$\text{div } \epsilon E = 0 \quad (4)$$

$$\lambda = \sigma + i\omega\epsilon/4\pi. \quad (5)$$

Consider two independent distributions of impressed forces $E_0'$, $E_0''$, and corresponding total currents $C'$ and $C''$ ($C = \lambda(E + E_0)$). Multiply (1) by $C = c \text{ curl } H/4\pi$ of the other distribution and subtract; then
Reciprocity in Electromagnetic and Other Systems

\[ E_0'C'' - E_0''C' = \frac{1}{4\pi} (E'' \nabla \times H' - E' \nabla \times H''). \]  

(6)

Remembering that

\[ E'' \nabla \times H' = \nabla \cdot (H' \times E'') + H' \nabla \cdot E'' \]

\[ - E' \nabla \times H'' = - \nabla \cdot (H'' \times E') - H'' \nabla \cdot E'. \]

(7)

Substituting values of \( \nabla \times E \) from (2) we have after some cancellation:

\[ E_0'C'' - E_0''C' = \frac{1}{4\pi} \left[ \nabla \cdot (E_1' \times H'') - \nabla \cdot (E'' \times H') \right]. \]  

(8)

This is a point relation. Upon integrating over a region of space and using the divergence theorem, the result may be formally summarized in the following theorem.

**Electromagnetic Reciprocity Theorem**

If \( E_0' \) and \( E_0'' \) are two independent distributions of impressed electric force which act on the total current (convection, conduction, polarization, and displacement current in the ether), and \( C' \) and \( C'' \) represent the total currents \[ = (\sigma + i\omega/4\pi)(E+E_0) \] resulting respectively from the action of \( E_0' \) and \( E_0'' \); if all quantities vary as \( e^{i\omega t} \), and if the properties of the medium \( \epsilon, \mu, \sigma \) are scalars and independent of the field vectors, then:

\[ \int \int \int [E_0'C'' - E_0''C'] dv = \frac{1}{4\pi} \int \int \int [E' \times H'' - E'' \times H'] \alpha ds \]  

(9)

where the surface integral extends over the boundary of the region of the volume integration.

It may be inferred that the surface integral represents the effect of sources (i.e., \( E_0C \)) beyond it, so that if there are no sources outside the region of integration the surface integral vanishes. For example, in Fig. 2 the region is shown bounded by a surface \( S_1 \) at infinity and a
closed surface $S_2$. It is demonstrated in Note 1 that the $S_1$-integral vanishes. Hence

$$
\int \int \int_R (\ )dv = - \int \int (\ )n ds \text{ (outward normal)} \tag{10}
$$

but $\int \int \int_{R+R'} (\ )dv = 0$, hence $\int \int \int_R (\ )dv = \int \int_{S_1} (\ )n ds$. This may be expressed in the following corollary:

**Corollary I**: If under the conditions of the above theorem, the volume integration extends over all space and $\mu - 1$, $\epsilon - 1$, $\sigma$, and $E_0$ are finitely distributed, then

$$
\int \int \int_{\infty} [E_0' C'' - E_0'' C']dv = \frac{c}{4 \pi} \int \int_{\infty} [E', H'' - E'' \times H'] n ds = 0, \tag{11}
$$

that is, the surface integral vanishes.

That this is true is almost evident by inspection when it is remembered that at a sufficient distance from a source the field vectors $E$ and $H$ become conjugate, that is, equal in magnitude and normal to each other and to the direction of propagation. To avoid interruption the detailed proof of this proposition is appended as Note 1 at the end of the paper.

The theorem in the form (9) involving both volume and surface integrals is more eloquent than the expressions of either Lorentz or Carson. Lorentz surrounded all sources with a surface integral; Carson used the volume integral over all space, assuming that all external influences could be accounted for by ponderomotive forces on the electricity. Theorem (9) combines both viewpoints. The surface integral is often useful in avoiding inquiries into complicated sources; it is simply necessary to surround the source by a surface and calculate $E \times H$ through it.

Equation (6) leads to another form. Since $\text{curl } H = 4\pi C/c$, $C$ being as before the total current;

$$
E_0' C'' - E_0'' C' = E_0'' C' - E' C'', \tag{12}
$$

and

$$
\int \int \int_{\infty} [E' C' - E' C''] dv = 0. \tag{13}
$$

This does not involve the impressed forces $E_0$ and is a general relation.

Considering for a moment the case contemplated in Carson's theorem, in which the impressed forces are of the nature of ponderomotive forces on the electricity and do not act on the displacement
current (which is equivalent to saying that there are no impressed charges), equation (1) would be written

\[ E_0 = \frac{c}{4\pi} \text{curl} \, H - \left( \lambda' + \frac{i\omega}{4\pi} \right) E \] (14)

where \( \lambda' = \sigma + i\omega(\epsilon - 1)/c \). We should then have Carson's result (Theorem II)

\[ \iint [E_0'u'' - E_0''u'] \, dv = 0. \] (15)

Here \( u \) represents the current due to the motion of electricity. When we have to deal with impressed currents \( u_0 \) due to the motion of electricity (1) may be written

\[ \text{curl} \, H = 4\pi\lambda E/c + 4\pi u_0, \] (16)

and the following reciprocal relations obtain

\[ \iint [E''u_0' - E'u_0''] \, dv = 0 \] (17)

note that at the field vector \( E \) replaces \( E_0 \) in this expression. An example of such an impressed current would be furnished by electrons acted upon by mechanical force.

If in (11) we substitute \( C = \lambda(E + E_0) \) then:

\[ \iint [E_0'C'' - E_0''C'] \, dv = \iint \lambda [E_0'E'' - E_0''E'] \, dv, \] (18)

which furnishes another useful relation involving only the \( E \)'s.

4. Example of the Application of the Electromagnetic Theorem. Calculation of the Effective Height of a Receiving Antenna. The physical significance of the reciprocity relations expressed by equation (11) may be brought out by means of a practical example. The following application to the calculation of the effective height of a receiving antenna is due to R. M. Wilmott\(^6\) and is simple and instructive.

The impressed voltage acting upon an element \( dx \) of the antenna (Fig. 3) is the component along the wire of the electric vector in the wave. Identify this with \( E_0 \) in (11) and call it \( E(x) \) where \( x \) is the distance along the antenna wire. Our problem is to find the value of a localized emf \( E \), which when inserted in some branch of the electric

circuit connected to the antenna terminals $AB$ will produce in this branch the same current which is produced by the action of the distributed force from the wave. The reciprocity theorem states that in any element $dx$ the current $I(x)$ produced by $E$ will be equal to the current at the point of application of $E$ due to an equal voltage $E(x)$ impressed at $dx$. Stated mathematically, for each element $dx$, $EI = E(x)I(x)$. This is true for each element of the antenna and is therefore true for the whole structure. Applying it to the whole antenna

$$E_0I = \int E(x)I(x) \cdot dx.$$  \hspace{1cm} (19)

We define the "effective height" as usual as $h = E/E'$; where $E'$ is the electric force in the wave. If $E(x)$ is uniformly distributed along the antenna, $E(x)/E' = 1$. The possibility of this calculation rests upon knowing $I(x)$. Wilmotte,\textsuperscript{7} in an important piece of experimental work, has verified in the case of an antenna of simple geometry, the approximate correctness of the old assumptions (Abraham, Pierce, etc.) of a sinusoidal distribution of current along the antenna.

In the case of a vertical antenna with $E$ at the base, $I(x) = I \sin \frac{2\pi x}{\lambda} / \sin \frac{2\pi l}{\lambda}$, and

$$h = \frac{1}{\sin \frac{2\pi l}{\lambda}} \int_0^l \sin \frac{2\pi x}{\lambda} \cdot dx = \frac{4n_0}{\pi} \frac{\sin^2 \pi n/4n_0}{\sin \pi n/2},$$  \hspace{1cm} (20)

where $n_0$ is the fundamental frequency and $n$ the operating frequency. At the fundamental $n = n_0$ and $h = 2l/\pi$.

An interesting case is that of the short rod antenna proposed by F. H. Drake\textsuperscript{8} for reception on airplanes in connection with a highly sensitive receiver. When the airplane is in horizontal flight the horizontal fusilage members contribute nothing, so that since $n = n_0$ the effective height is equal to half the actual height.

\textsuperscript{7} R. M. Wilmotte, Jour. I. E. E., 66, 617, 1928.
\textsuperscript{8} F. H. Drake, Contributions from the Radio Frequency Laboratories. No. 8, October 1928; Proc. I. R. E., 17, 306; February, 1929; Aero Digest, October, 1928.
5. Mechanical Systems. By a mechanical system is meant a system composed of moving parts having mass, compliance, and resistance (dissipation), these parts being usually amenable to direct physical observation. Elastic solids, fluids and gases are really also mechanical systems, but for convenience will be treated as acoustical. The reciprocal relations in a mechanical or dynamical system which can be described by means of linear equations have been discussed by the late Lord Rayleigh. Reciprocity exists between the displacements and forces, although it is often convenient, especially when employing electric-circuit analogies of the mechanical system, to replace the displacements by the velocities.

If a set of impressed forces $F_1', F_2', F_3' \cdots$ produce displacements $x_1', x_2', x_3' \cdots$, and another independent set of forces $F_1'', F_2'', F_3'' \cdots$ produce displacements $x_1'', x_2'', x_3'' \cdots$, then

$$F_1'x_1'' + F_2'x_2'' + \cdots = F_1''x_1' + F_2''x_2' + \cdots,$$

(21)
or generally

$$\sum_n (F_n'x_n'' - F_n''x_n') = 0.$$  

(22)

If we are dealing with velocities the $x$'s in (22) are simply replaced by $v$'s.

The circumstances of validity in the mechanical system are the same as those for an electrical system as discussed in Sect. 3, viz. quasi-stationary state with quantities varying as $e^{i\omega t}$, linearity in the relations between forces and displacements, and complete reversibility. Reversibility should not be confused with friction. Failure of reciprocity would follow if elastic hysteresis or an equivalent phenomenon were present. Friction will also introduce non-linearity when the velocity is small. All these practical limitations must be taken into consideration in making applications of the theoretical theorems.

6. Electro-Mechanical Systems. Since there are reciprocal relations between impressed electric forces and currents in an electric system, and between impressed forces and velocities in a mechanical system, they can be shown to hold in an interconnected electro-mechanical system, provided that the interconnections are linear and reversible.

In considering such composite systems I have found the use of an interaction, or transduction coefficient convenient. This may be defined as follows: let $F$ be the mechanical force acting upon an element of a mechanical system as the result of a current $I$ flowing in an element of

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an electrical system. Then the *electro-mechanical transduction* $T_{em}$ between these elements is defined as

$$F_0 = T_{em} I.$$  

Similarly if $E_0$ represents the electromotive force due to an element of a mechanical system moving with velocity $v$, then the *mechano-electrical transduction* $T_{me}$ is defined by

$$E_0 = T_{me} v.$$  

If the transformation is reversible $T_{em} = T_{me}$; and if the interaction is linear $T$ is independent of $I$ or $v$. The dimensions of $T$ are $[M^{1/2}L^{-3/2}]$. The "mutual energy" is equal to $1/2 TIv$.

This concept is of considerable utility. Recent years have seen an increasing use of electrical analogues of mechanical systems. By this means we can bring to bear upon mechanical problems a great deal of knowledge concerning electrical networks which has resulted from the economic necessity for studies of electrical problems. In such formulations the role of the transduction coefficient is something like that of a mutual inductance between the real electrical circuit and the electrical analogue of the mechanical system.

As a simple example of this point of view consider the electro-mechanical system shown in Fig. 4. The coil is situated in a uniform magnetic field, $H_0$. We shall neglect eddy currents. The ponderomotive force in each element of volume is, generally,

$$F_0 = \frac{1}{c} [I \times B] - \frac{1}{8 \pi} H^2 \text{ grad } \mu. \quad (23)$$

In the above case if the coil is of copper $B = H, \mu = 1$, so that

$$F_0 = \frac{1}{c} [I \times H_0] \quad (24)$$
on each element of the coil. Hence \( T_{em} = H_0/c \). In the mechano-electrical process the induced emf due to a motion of an element of the wire with velocity \( v \) is

\[
E_0 = \frac{1}{c} [v \times H_0]
\]

so that \( T_{me} = H_0/c \). The processes are thus reversible and the transductance is symmetrical, \( T_{me} = T_{em} \).

The subject of units deserves some attention. If \( E_0 \) is expressed in volts, \( I \) in amperes, \( F \) in dynes, and \( v \) in cm per sec., then

\[
F_0 = 10^{-1} I H_0 \text{ dynes amp}
\]

\[
E_0 = 10^{-8} v H_0 \text{ volts, cm/sec}
\]

therefore \( T_{em} = 10^{-1} \); \( T_{me} = 10^{-8} \). \( T_{me} \) can be adjusted to have the same numerical value as \( T_{em} \) when \( E \) and \( I \) are expressed in practical units by expressing \( F \) and \( v \) in appropriate units. We shall call these units in this particular case mechanical volts and mechanical amperes, respectively, or alternatively, practical electric units of mechanical force and velocity. A collection of the principal electro-mechanical analogues and of conversion factors from c.g.s. mechanical units to “practical electric” mechanical units is given in Table I.

**TABLE I**

<table>
<thead>
<tr>
<th>Mechanical Quantity</th>
<th>Symbol</th>
<th>Definition</th>
<th>Electrical Analogue</th>
<th>c.g.s. Unit</th>
<th>Value in Practical Electrical Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>displacement</td>
<td>( x )</td>
<td>charge</td>
<td>charge</td>
<td>1 cm</td>
<td>( 10^{-4} ) mech. coulombs</td>
</tr>
<tr>
<td>force</td>
<td>( F )</td>
<td>voltage</td>
<td>voltage</td>
<td>1 dyne</td>
<td>( 10^{-4} ) mech. volts</td>
</tr>
<tr>
<td>velocity</td>
<td>( v )</td>
<td>current</td>
<td>current</td>
<td>1 cm/sec</td>
<td>( 10^{-4} ) mech. amps.</td>
</tr>
<tr>
<td>energy</td>
<td>( W )</td>
<td>energy</td>
<td>energy</td>
<td>1 erg</td>
<td>( 10^{-7} ) joules.</td>
</tr>
<tr>
<td>power</td>
<td>( P )</td>
<td>power</td>
<td>power</td>
<td>1 erg/sec</td>
<td>( 10^{-7} ) watts.</td>
</tr>
<tr>
<td>resistance</td>
<td>( R )</td>
<td>resistance</td>
<td>resistance</td>
<td>1 dyne/cm/sec</td>
<td>( 10^{-9} ) mech. ohms.</td>
</tr>
<tr>
<td>mass</td>
<td>( m )</td>
<td>inductance</td>
<td>inductance</td>
<td>1 gram</td>
<td>( 10^{-9} ) mech. henrys.</td>
</tr>
<tr>
<td>compliance</td>
<td>( C )</td>
<td>capacitance</td>
<td>capacitance</td>
<td>1 cm/dyne</td>
<td>( 10^{-9} ) mech. farads</td>
</tr>
<tr>
<td>stiffness</td>
<td>( s )</td>
<td></td>
<td></td>
<td>1 dyne/cm</td>
<td></td>
</tr>
</tbody>
</table>

According to this point of view the system in Fig. 4 may be represented as shown in Fig. 5. The transduction is represented by \( T \), and symbolically as a mutual inductance. In this “mutual inductance” representation only the inductance directly concerned with the interaction is included in the primary coil and only the masses of the mechanical elements associated with the transfer are included in the secondary “coil.” The so-called *motional impedance* of the system is easily seen to be the impedance between the electrical terminals \( AB \), and the *damped impedance* that between \( AB \) with the secondary open.
This representation in familiar electric form enables us to see at a glance what reactions the mechanical system will have upon the electrical system, and conversely the effect of the electrical circuit upon the motion of the mechanical system when an impressed mechanical force is acting. The latter effect is usually neglected by writers on this subject but obviously becomes of importance when the transfer is high. The conditions for maximum transfer between the systems can also be immediately formulated.

The demonstration of the reciprocal relations in the interconnected system now proceeds along the usual lines. Let a set of impressed electric forces \( E_1', E_2' \ldots E_n' \) act in the electrical system, and a set of mechanical forces \( F_1', F_2', \ldots, F_n' \) act in the mechanical system; let the corresponding currents in the electrical system be \( I_1', I_2', \ldots, I_n' \) and the velocities in the mechanical system be \( v_1', v_2', \ldots \); let also \( F'', E'', I'', v'' \) represent a second set of such forces, currents and velocities. All quantities vary as \( e^{iut} \).

\[
\begin{align*}
E_1 &= I_1 z_1 + I_2 m_{12} + I_3 m_{13} + \cdots v_1 T_{11} + v_2 T_{12} + \cdots \\
E_2 &= I_1 m_{12} + I_2 z_2 + I_3 m_{23} + \cdots v_1 T_{21} + v_2 T_{22} + \cdots \\
&\vdots \\
F_1 &= I_1 T_{11}' + I_2 T_{12}' + I_3 T_{13}' + \cdots v_1 Z_1 + v_2 M_{12} + \cdots \\
F_2 &= I_1 T_{21}' + I_2 T_{22}' + I_3 T_{23}' + \cdots v_1 M_{21} + v_2 Z_2 + \cdots
\end{align*}
\]

Here \( T_{jk} \) represents the electric force in mesh \( j \) due to velocity \( v_k \) of mechanical mesh \( k \), and \( T_{jk}' \) represents the mechanical force in mechanical mesh \( j \) due to the electric current in mesh \( k \); \( z \) and \( m \) represent electrical self and mutual impedances, and \( Z \) and \( M \) represent corresponding mechanical impedances.

Set up another system of equations for the other set of forces \( E'' \) and \( F'' \). Multiply line 1 of (27) by \( I_1'' \), line 2 by \( I_2'' \) and so on, and line 1 of the second set by \( I_1' \), etc., and subtract. Then if the connections
are perfectly reversible, that is \( M_{jk} = M_{kj}; m_{jk} = m_{kj}; \) and \( T_{jk} = T'_{kj}, \) we obtain the following general reciprocal relations

\[
\sum_n E_n' I_n'' + F_n' v_n'' = \sum_n E_n'' I_n' + F_n'' v_n' .
\]  

(28)

Considering the simple case in which all the forces except two are zero

\[ E'I'' = F''G', \]  

(29)

which is an expression of simple reciprocity. In words this states that: if in an interconnected electro-mechanical system a unit electric force impressed in an electrical mesh produces a certain velocity in a mechanical mesh, then an equal mechanical force acting in the mechanical mesh will produce a current in the electrical mesh which is numerically the same as the velocity previously produced in the mechanical mesh.

Equation (28) is correct as it stands if \( F \) is expressed in dynes, \( v \) in cm per sec., and \( E \) and \( I \) are expressed in electromagnetic c.g.s. units, or electrostatic c.g.s. units. In order to be correct when \( E \) and \( I \) are expressed in volts and amperes, \( F \) must be expressed in mechanical volts and \( v \) in mechanical amperes as above. If \( E \) and \( I \) are in volts and amperes, and \( F \) and \( v \) are in dynes and cm per sec., then a factor of \( 10^7 \) will occur, that is

\[ E'I'' = 10^7F''v'. \]  

(30)

In the more general expression (28) this factor will also precede \( Fv \) whenever they occur.

A word may be said as to the linearity and reversibility in practical interconnections between electrical and mechanical systems. In the example of Fig. 4 reversibility may be safely assumed, but in general when the system contains ferro-magnetic materials, hysteresis and the non-linearity of the \( B-H \) relation demand some caution. When the amplitudes are small the higher order effects can generally be ignored. Eddy currents not depending upon magnetic behavior will not disturb the reversibility or linearity.

7. Acoustical Systems. Turning now to the propagation of sound of small amplitude in air, it will be convenient for reference purposes to preface the discussion of the reciprocal relations by a brief statement of the principal equations. These will be given in modern vector form and the excess pressure \( p \) and velocity \( v \) will be regarded as the fundamental quantities, analogous to \( E \) and \( H \) of the electromagnetic field and \( F \) and \( v \) of the mechanical system.

If we ignore viscosity and have a ponderomotive force \( F_o \) per unit mass acting upon the body of the gas, the motion may be described by Euler's equation
\[
F_0 = \frac{\partial v}{\partial t} + \frac{1}{2} \text{grad } v^2 - v \times \text{curl } v + \frac{1}{\rho} \text{grad } P. \tag{31}
\]

In addition we have the equation of continuity
\[
\frac{\partial \rho}{\partial t} + \rho \text{div } v + v \text{grad } \rho = 0. \tag{32}
\]

If the motion is irrotational, \(\text{curl } v = 0\), and if the amplitudes are small we may neglect the terms \(\text{grad } (v \cdot v)\) in (31) and \(v \cdot \text{grad } \rho\) in (32); so that
\[
F_0 = \frac{\partial v}{\partial t} + \text{grad } P/\rho; \tag{33}
\]
\[
\frac{\partial \rho}{\partial t} + \rho \text{div } v = 0. \tag{34}
\]

A further relation between \(\rho\) and \(P\) is furnished by considering the adiabatic expansion and compression of the gas
\[
P/P_0 = (\rho/\rho_0)^\gamma, \tag{35}
\]
where for air \(\gamma = 1.4\), and \(P_0\) and \(\rho_0\) represent the steady values of these quantities. Now eliminate \(P\) from (31) and (32) by means of (35) and let the excess pressure \(p = P - P_0\); then
\[
F_0 = \frac{\partial v}{\partial t} + \text{grad } p/\rho_0, \tag{36}
\]
\[
\text{div } v = -\frac{1}{\gamma P_0} \frac{\partial p}{\partial t}. \tag{37}
\]

These equations and the boundary conditions are sufficient to determine \(p\) and \(v\). Take the divergence of (36)
\[
\text{div } F_0 = \frac{\partial}{\partial t} \text{div } v + \frac{1}{\rho_0} \nabla^2 p = -\frac{1}{\gamma P_0} \frac{\partial^2 p}{\partial t^2} + \frac{1}{\rho_0} \nabla^2 p_0, \tag{38}
\]
from (37) so that \(p\) satisfies the wave-equation
\[
\nabla^2 p - \frac{\rho_0}{\gamma P_0} \frac{\partial^2 p}{\partial t^2} = \rho_0 \text{div } F_0. \tag{37'}
\]

Similarly, taking the gradient of (37)
\[
\text{grad } \text{div } v = \nabla^2 v - \frac{1}{\gamma P_0} \frac{\partial}{\partial t} \text{grad } p = \frac{\rho_0}{\gamma P_0} \left( \frac{\partial^2 v}{\partial t^2} - \frac{\partial F_0}{\partial t} \right), \tag{40}
\]
so that \(v\) also satisfies the wave-equation
\[
\nabla^2 v - \frac{\rho_0}{\gamma P_0} \frac{\partial^2 v}{\partial t^2} = -\frac{\rho_0}{\gamma P_0} \frac{\partial F_0}{\partial t}. \tag{41}
\]
The energy relations involving these quantities are interesting. Multiply (36) by \(v p_0\) to get the rate at which work is done by the impressed force \(F_0\) per unit volume:

\[
\rho_0 F_0 v = \rho_0 v \frac{\partial v}{\partial t} + v \text{ grad } p.
\]  

But \(v \cdot \text{ grad } p = \text{ div } (v \cdot p) - p \text{ div } v\). Substituting the value of \(\text{ div } v\) from (37) and integrating over a region of space

\[
\int \int \int_{(1)} \rho_0 F_0 v d\nu = \frac{\partial}{\partial t} \int \int \int_{(2)} \left[ \frac{1}{2} \rho_0 v^2 + \frac{1}{2} \frac{1}{\gamma} \frac{1}{p^2} \right] d\nu \]

\[
+ \int \int_{(3)} p \cdot v_n d\sigma.
\]

This states that the rate at which work is done by the impressed forces (1) is equal to the rate of increase of the kinetic energy (2) plus the rate of increase of the energy of compression (3) plus the rate at which energy is escaping across the boundary (4). The flow of acoustic power is represented by the vector \(P_v\), which is therefore analogous to the Poynting-vector \(E \times H\) of the electromagnetic theory.

Reciprocal relations in acoustics have been stated by Helmholtz.\(^{10}\) We shall, however, derive the relations anew and in somewhat greater generality in terms of the quantities \(F, p,\) and \(v\) which are here regarded as fundamental, rather than in terms of the "sources" and velocity-potential of the historical treatment.

Consider two independent sets of impressed forces \(F_0', F_0''\), pressures \(p', p''\) and velocities \(v', v''\). Multiply (36) by the \(v\) of the other set and subtract

\[
F_0' v'' - F_0'' v' = \frac{1}{\rho_0} (v'' \text{ grad } p' - v' \text{ grad } p'').
\]  

in the quasi-stationary state (all quantities varying as \(e^{i\omega t}\)). But

\[
v \text{ grad } p = \text{ div } (vp) - p \text{ div } v;
\]

hence, substituting the value of \(\text{ div } v\) from (37),

\[
\rho_0 (F_0' v'' - F_0'' v') = \text{ div } (v'' p' - v' p'').
\]  

This is a point relation. The result of integration over a region of space with the employment of the divergence theorem may be expressed in the following theorem.

Acoustical Reciprocity Theorem

If in an acoustical system comprising a medium of uniform density \( \rho_0 \) propagating irrotational vibrations of small amplitude, a distribution of impressed ponderomotive force \( F'_0 \) per unit mass per unit volume varying as \( e^{i\omega t} \) produces a velocity \( v' \) and pressure \( p' \), and a second distribution of impressed force \( F''_0 \) produces velocity \( v'' \) and pressure \( p'' \), then

\[
\rho_0 \iint_\Omega (F'_0 v'' - F''_0 v') \, dv = \iint_\Omega (v'' p' - v' p'') \, ds, \tag{46}
\]

where the surface integral is to be taken over the boundaries of the region of volume integration.

**Corollary I:** If the volume integral extends over all space and the distributions are finite:

\[
\iint \int_{\text{all space}} (F'_0 v'' - F''_0 v') \, dv = 0. \tag{47}
\]

The surface integral, as in the electrical theorem, represents the effect of impressed forces or other sources outside the region of volume integration, or those introduced at the boundaries. If there are no such sources the surface integral vanishes. This proposition is demonstrated in Note 2 at the end of the paper. Either the volume or the surface integral may be used, the surface integral being then taken over surfaces enclosing all sources.

The irrotational waves considered above are of a simple "compressional" type.

Body forces of the type \( F_0 \) postulated above are conceivable but are seldom encountered in practical acoustical problems. What we are generally concerned with are boundary forces arising from the motion of a mechanical boundary or discontinuity. The transfer of energy to the acoustical system may also be achieved by a thermodynamic process, as in the thermophone. An example of the \( F_0 \) type of body force would be the direct electrical excitation of an ionized gas; an example of the boundary type of force would be provided by a moving diaphragm, or piston. In this case the reciprocal relations are exhibited by the surface integral, that is

\[
\iint_{\text{surface}} (p' v'' - p'' v') \, ds = 0. \tag{48}
\]

Either \( p \) or \( v \) may be regarded as "impressed." In the case of the radi-
ation of sound from a vibrating piston, for example, either the impressed mechanical force on the piston or its velocity may be specified.

The above demonstration relates to sound propagation in a gas and should be supplemented by a similar demonstration for elastic solid and liquid media since the gas system may have such boundaries. The equations and the proof for irrotational vibrations in these media are the same as those given if the compressibility of the gas \((1/\gamma P_0)\) is replaced by the reciprocal of the bulk modulus \(1/k\) for the solid or liquid, or \(1/(\lambda + 2\mu)\) for the solid.

It may be remarked in passing that a special type of force is required to set up pure rotational waves (transverse or shear waves) in an isotropic elastic solid, and in general this type of vibration will not be excited in the solid by an incident irrotational wave from the gas.

8. Mechano-Acoustical Systems. Let us now consider the reciprocal relations in interconnected mechanical and acoustical systems. A mechanically driven piston is an example of such interconnection. This sort of transduction differs from that between an electrical circuit and a mechanical system in that the mechanical and acoustic systems are really both mechanical. It is, in fact, more nearly analogous to the interconnection of an electrical network and an electrical space-

![Fig. 6](image)

radiation or receiving system, both of these processes being electrical. The forces in the acoustic system are mechanical forces, and the velocities are physical velocities, and both are measured in the same system of units which is used for the corresponding mechanical quantities. An extended argument is therefore not necessary to establish reciprocity in the connected system, and it is merely necessary to identify \(p\) and \(v\) of (46) with the force and velocity of the mechanical system. The acoustic pressure is the force per unit area of the mechanical system, and the velocities must be the same.

As an example of reciprocity in such a system consider two diaphragms \(A\) and \(B\) (Fig. 6). If we impress on a unit area at \(P_1\) a force \(F_0'\), the velocity \(v'\) produced at a point \(P_2\) of the other diaphragm will be equal to the velocity \(v''\) which would be produced at \(P_1\) if an equal force \(F''\) were impressed on unit area at \(P_2\). An important use of this theorem
is the deduction of the receiving properties of a sound transducer from its transmitting properties, and vice versa.

9. Interconnected Electrical, Mechanical and Acoustical Systems. Reciprocal relations have been shown to hold in an electro-mechanical system and in a mechano-acoustical system, therefore they hold in a composite system involving all three systems in interconnection in this order. (Reversible transduction directly between electric and acoustical systems is rare and a mechanical system will usually intervene between them.) The reciprocity in such a composite electric-mechanic-acoustic system as shown in Fig. 7 may be expressed as follows, $F_0$ being an acoustical body-force

$$
\int \int \int \rho_0 F_0'' v' w = E'I''.
$$

(49)

If we deal with pressure in the sound field, $p''$ will replace $\rho_0 F_0''$. In words: if a generator of emf $E'$ (in em or es c.g.s. units) operates in an electrophone circuit and produces at a point $P$ in the sound field a velocity $v'$, then a numerically equal force in dynes exerted at $P$ on the gas material per unit volume, or communicated as a pressure, will produce a current $I''$ in the electrical mesh equal to the previously produced velocity $v'$ at $P$.

The transduction must of course be reversible. An electromagnetic or electrostatic device (e.g., condenser microphone) would probably be sufficiently reversible, but a thermophone would certainly violate the requirements because of the irreversibility of the Joulean effect.

As a further extension of the composite system we may set up two reversible electrophones, as shown in Fig. 8, and state that in the terminating electrical circuits
The reciprocal relations are then the same as if the system were entirely electrical.

10. Example of the Application of the Extended Reciprocity Relations. Frequency Calibration of an Electrophone. The above extended reciprocity relations between two reversible electrophones has an interesting and useful application in a method for determining the frequency response characteristics of an emitting electrophone or a receiving microphone. This may be called the "method of three electrophones" and is analogous to the method of measuring the resistance of a ground by the method of three grounds, or of the effective height of an antenna by using two additional antennas. The frequency characteristic of the electrophone usually desired is that in a certain direction, say normal to the sound-emitting area. (Characteristics in other directions may obviously be obtained in the same way.)

The electrophones are used two at a time (Fig. 9) and are situated so far apart that the wave-front is substantially plane and the interaction between them is negligible. The purpose of this is to obtain data which will be fundamentally related to plane waves; investigations based on divergent waves may be carried out by proper adjustments of separation. One unit is excited by a generator $E'$ of variable frequency and the current (or equivalent) in the other is measured. Let the frequency characteristics of the individual electrophones with respect to plane waves be denoted by $f_1(\omega)$, $f_2(\omega)$ and $f_3(\omega)$, respectively. Then for transmission from No. 1 to No. 2, for example, the measurement will give $F_{12}(\omega) = f_1(\omega)f_2(\omega)$. The transmission characteristic of No. 2 to No. 3 ($F_{23}$) is next obtained, and then that from No. 1 to No. 3 ($F_{13}$). This experimental information suffices to determine $f_1$, $f_2$, or $f_3$. For

$$F_{12}(\omega) = f_1(\omega)f_2(\omega)$$
$$F_{23}(\omega) = f_2(\omega)f_3(\omega)$$
$$F_{13}(\omega) = f_1(\omega)f_3(\omega),$$

(51)
which is a set of simultaneous functional equations to be solved for the $f$'s. The solutions are

$$ f_1(\omega) = (F_{12}F_{13}/F_{23})^{1/2} $$
$$ f_2(\omega) = (F_{12}F_{23}/F_{13})^{1/2} $$
$$ f_3(\omega) = (F_{12}F_{23}/F_{13})^{1/2}. $$

If absolute values for the $f$'s are desired care must be taken to measure properly the $F$'s for each pair. For most purposes, however, all that is required is a curve of the relative variations of $f(\omega)$ with frequency; in this case absolute values of the $F$'s are of no importance.

It may be noted that in the special case of two electrophones having identical frequency characteristics it is merely necessary to measure the frequency-transmission $F$ for this pair since $F_{12}(\omega) = f_{12}(\omega)f_{21}(\omega)$ and $f_1 = (F_{12})^{1/2}$. This match will rarely be sufficiently exact in practical apparatus, except perhaps with precision microphones of the air-damped variety which have been accurately made and adjusted as to membrane tension.

It is also worth noticing that this method of calibration is equivalent to a Rayleigh-disk calibration in that it also takes into account the effect of reflection by the microphone.\(^{11}\)

As will be seen from Fig. 9 this application of reciprocity requires only one of the electrophones (No. 2) to be reversible. Also note that one of them, No. 1, is always used as a source, and that the other, No. 3, is always used as a receiver. Thus a thermophone (irreversible) might be used at No. 1 as a source, with a reversible electrophone No. 2 as a comparator, to calibrate a carbon microphone, No. 3 (irreversible). It is needless to say that in order to obtain fundamental data the work should be performed in the open air and at a sufficient height to avoid disturbance due to reflection from the earth.

**Note 1**

It is required to evaluate the integral

$$ \int \int \int_{-\infty}^{\infty} [E_0' - E_0'' C']dv. $$

To do this we shall make use of the electromagnetic potentials $A$ and $\phi$, between which and the field vectors the following well-known relations exist

\(^{11}\) For an account of this effect and of another method for evaluating it, see Stuart Ballantine, *Contributions from the Radio Frequency Laboratories*, No. 9, December 1928; *Phys. Rev.*, 32, 988, 1928; *Proc. I. R. E.*, 16, 1639; December, 1928.
\( B = \text{curl} \ A; \quad E = -\text{grad} \ \phi - i\omega A/c. \)  

(2)

also
\[
\Delta^2 A - \left( \frac{i\omega}{c} \right)^2 A = \text{dal} \ A = -i\omega E_0/c - 4\pi \mu/c - \text{curl} \ H
\]

(3)

\( \text{dal} \ \phi = -4\pi \rho \)  

(4)

\( \text{div} \ A = -i\omega \phi/c \)  

(5)

\[ M = (\mu - 1)H; \quad B = \mu H \]

(6)

\( \mu \) is the current due to the motion of electricity; \( \rho \) is the charge (exclusive of impressed charge, \( \rho_0 = \text{div} \ E_0/4\pi \)). Equation (6) of Sect. 3 may be written, since \( \text{curl} \ H = \text{curl} \ B - \text{curl} \ M \):

\[
E'_0C'' - E''_0C' = \frac{c}{4\pi} (E'' \text{ curl} \ B' - E' \text{ curl} \ B''

- E' \text{ curl} M' + E' \text{ curl} M''). \quad (7)
\]

Substituting for \( E \) and \( B \) their values (2) in terms of the potentials, we have

\[
E'_0C'' - E''_0C' =
\]

(1) \( \text{grad} \ \phi'' \ \text{dal} \ A' - \text{grad} \ \phi' \ \text{dal} \ A'' \)

(2) \( + \frac{i\omega}{c} (A''\Delta^2 A' - A'\Delta^2 A'') \)

(3) \( + \text{grad} \ \phi'' \ \text{curl} \ M' - \text{grad} \ \phi' \ \text{curl} \ M'' \)

(4) \( + \frac{i\omega}{c} (A'' \text{ curl} M' - A' \text{ curl} M''). \quad (8)
\]

We will consider these terms in order. Substitute in (1) the value of \( \text{dal} \ A \) from eq. (3), then

\[
(1) = -\text{grad} \ \phi'' \left( \frac{4\pi u'}{c} + \frac{i\omega E_0'}{c} \right) + \text{grad} \ \phi' \left( \frac{4\pi u'}{c} + \frac{i\omega E_0'}{c} \right) \quad (a)
\]

\( -\text{grad} \ \phi'' \ \text{curl} \ M' + \text{grad} \ \phi' \ \text{curl} \ M'''. \)  

(b)

Note that term (1b) cancels term (3). Transform (a) by means of the theorem

\[
\iint \iint D \ \text{grad} \ \psi \, dv = \iint \iint \text{div} \ (\psi D) \, dv - \iint \iint \psi \ \text{div} \ D \cdot dv
\]
and since \( \text{div } C = 0 \)

\[
(1a) \quad = -\frac{i \omega}{c} \int \int \int (\phi'' \text{ div } E' - \phi' \text{ div } E')dv
\]

\[ + \frac{i}{c} \int \int \int [\phi'(4\pi u'' + i\omega E_0'') - \phi''(4\pi u' + i\omega E_0')]\,ds. \]

If the integration embraces all space the integral over the infinitely distant surface vanishes for a finite distribution of \( E_0 \) and \( u \). As to the volume integral, note that \( \text{div } E = 4\pi \rho \), the value of which is given by (4) above. Making this substitution and applying Green's theorem

\[
(1a) \quad = \frac{i \omega}{c} \int \int \int (\phi'' \Delta^2 \phi' - \phi' \Delta^2 \phi'')dv
\]

\[ = \frac{i \omega}{c} \int \int \left( \phi'' \frac{\partial \phi'}{\partial n} - \phi' \frac{\partial \phi''}{\partial n} \right) \,ds. \]

If the surface is at infinity, \( \partial/\partial n = -i \omega/c \) so that this term also vanishes. Terms (1) + (3) are therefore = 0.

Term (2) vanishes for similar reasons. Applying Green's theorem

\[
(2) \quad = \frac{i \omega}{c} \int \int \int (A'' \Delta^2 A' - A' \Delta^2 A'')dv
\]

\[ = \int \int \left( \frac{\partial A'}{\partial n} - A' \frac{\partial A'}{\partial n} \right) \,ds. \]

This can be regarded as a scalar equation for each component of \( A \).

As before, \( \partial/\partial n = -i \omega/c \) for the infinitely distant surface so that the integral over all space is zero.

The remaining term (4) may be transformed by means of the theorem

\[
\int \int \int P \text{ curl } \Omega \,dv = -\int \int (P \times \Omega)\,n \,ds
\]

\[ + \int \int \int \Omega \text{ curl } P \,dv \]

so that

\[
(4) \quad = \frac{i \omega}{c} \int \int \int (M' \text{ curl } A'' - M'' \text{ curl } A')dv
\]

\[ - \int \int (A'' \times M' - A' \times M'')\,n \,ds. \]
If, as before, the distribution of $\mu - 1$ is finite the surface integral vanishes. As to the volume integral, since curl $A = B$ and $M = (\mu - 1) B/\mu$

$$= \frac{i\omega}{c} \int \int \int \frac{\mu - 1}{\mu} (B'B'' - B'B')dv = 0. \tag{4}$$

This completes the demonstration that the integral (1) vanishes when taken over all space.

**Note 2**

It is required to show that in the acoustical case

$$\rho_0 \int \int \int (F_0' v'' - F_0'' v')dv = \int \int \int (v'' p' - v' p'')_n ds \tag{1}$$

generally, and if the integration is over all space the result is equal to zero. Let there be two distributions of impressed force $F_0'$ and $F_0''$ of the type $e^{i\omega t}$. Then according to equation (49)

$$\Delta^2 v' - k^2 v' = A F_0' \tag{2}$$

$$\Delta^2 v'' - k^2 v'' = A F_0'' \tag{3}$$

where $k^2 = -\rho_0 \omega^2 / \gamma P_0$ and $A = -i\omega_0 / \gamma P_0$. Multiply (2) by $v''$ and (3) by $v'$ and subtract

$$A (F_0' v'' - F_0'' v') = v'' \Delta^2 v' - v' \Delta^2 v''. \tag{4}$$

Integrate this over a region of space and transform the right-hand side by means of Green's theorem

$$A \int \int \int (v'' \Delta^2 v' - v' \Delta^2 v'')dv = \int \int \left( v'' \frac{\partial v'}{\partial n} - v' \frac{\partial v''}{\partial n} \right)_n ds. \tag{5}$$

If we consider an integral over all space and take the surface to be that of a sphere, $\partial / \partial n = -i\omega / c$, and the surface integral vanishes, giving

$$\int \int \int (F_0' v'' - F_0'' v')dv = 0. \tag{6}$$

The properties of the medium $P_0$ and $\rho_0$ have been assumed to be uniform. The theorem can also be proved for the case where they vary from point to point, but this paper is already long and this work will not be reproduced.